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OPTIMUM AND SUB-OPTIMUM CONSTANT FEEDBACK CONTROL OF LINEAR SAMPLED-DATA SYSTEMS

PHILLIP SHELDON HOWARD
and
THOMAS LEE WASMUND

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OPTIMUM AND SUBOPTIMUM CONSTANT FEEDBACK CONTROL OF LINEAR SAMPLED-DATA SYSTEMS

by

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ABSTRACT

In this paper constant feedback control laws are investigated for linear sampled-data systems. Derivations for the minimum-time problem and the infinite stage regular problem which yield constant feedback optimal control laws are presented. A case study for the finite stage regulator problem indicates that constant feedback control, although suboptimal, may be feasible from a design viewpoint. Several numerical examples illustrate the results.

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1. INTRODUCTION

1.1 General

Optimum performance implies that a system performs in the "best" possible manner according to some specified set of criteria. If a criterion is established in a mathematical form, called a performance index, the "best" performance results when this index is minimized or maximized. An optimum controller then generates the sequence of control signals necessary to obtain an extremum of the performance index.

This thesis investigates the optimal synthesis of linear sampled-data control systems in order to determine an optimum or "suboptimum" control law in the form of stationary, linear feedback of the plant variables.

If the performance index is specified as a quadratic functional on the plant variables and the control variables, for a linear system, the optimum control law that is derived to obtain an extremum of this functional generally results in a linear, time-varying function of the plant variables, i.e., multiloop feedback. The physical realization of a time-varying control law requires a digital computer, or some other time-varying device. If a stationary control law can be obtained, the complexity of design, weight and cost of the controller is reduced. This is the motivation for this investigation.

It will be shown in two cases that the optimum control law is stationary. In a third case the optimum control law is time-varying, but a case study shows that a stationary control law can be obtained that is "close" to optimum in performance.

1.2 System Description

The physical system to be controlled is called the plant, and is described by a set of linear differential equations with constant coefficients. It is controlled by certain physical quantities, called inputs or controls. The physical variables of the plant which can be measured directly are called outputs.

By definition, the state of a dynamic system is the smallest collection of numbers which must be specified at time $t=t_o$, along with the control for time $t \geq t_o$, in order to be able to predict the behavior of the system for any time $t \geq t_o$. In explanation, suppose the plant, which is described by an nth order differential equation, is simulated by an analog computer. This requires exactly n integrators. The outputs $x_1,\ldots,x_n$ of the n integrators, measured at any instant of time, are the state of the plant at that instant. The state may be regarded as a vector (nx1 matrix) and is written as

$$\underline{x}(t) = \begin{bmatrix} x_1(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix}$$

where the components $x_1(t),\ldots,x_n(t)$ are the state variables.

The plant can be described by the state variable form of the differential equation

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t) \tag{1-1}$$

and the output equation

$$\underline{y}(t) = C\underline{x}(t); \tag{1-2}$$

or, as used in this discussion, by a vector matrix difference equation, the discrete time solution of equation (1-1)

$$x((k+1)T) = \phi(T)\underline{x}(kT) + D(T)\underline{u}(kT) \tag{1-1a}$$

and the output equation

$$\underline{y}(kT) = C\underline{x}(kT). \tag{1-2a}$$

Here $\underline{u}$ is an m dimensional vector $(m \leq n)$ denoting the control inputs to the plant; $\underline{y}$ is a p dimensional vector $(p \leq n)$ denoting the outputs of the plant; A and ϕ are constant (nxn) matrices; B and D are constant (nxm) matrices; C is a constant (pxn) matrix.

The methods required for the evaluation of the transition matrix, ϕ , and the matrix D, from the matrices A and B, and the development of equations (1-1a) and (1-2a) from (1-1) and (1-2), are given in Appendix I.

It is assumed that the control is exerted on the plant by means of a piecewise constant signal whose level can change only at a regularly spaced "sampling points" or "sampling instants". Thus,

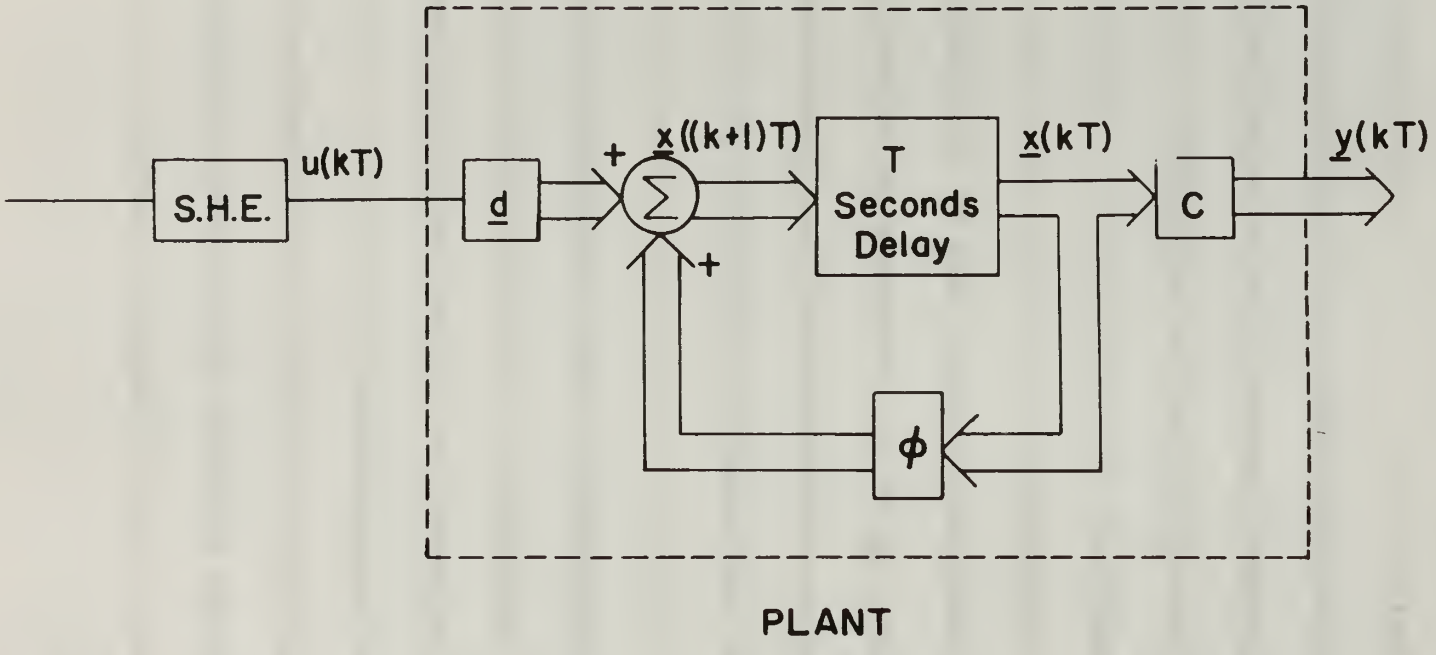

Fig. I-I BLOCK DIAGRAM OF THE PLANT

$$\underline{u}^*(t) = \underline{u}(kT); \quad kT \le t < (k+1)T$$
$$k = 0,1,2,\ldots \tag{1-3}$$

where T is a positive constant called the sampling period; $t = kT$ is the kth sampling point.

It is customary to symbolize the sampling operation by means of a special element, the sample and hold element (S.H.E.), which converts the continuous signal $\underline{u}(t)$ into a sampled signal $\underline{u}^*(t)$ as given by equation (1-3). The difference equation (1-1a) relates the state $\underline{x}((k+1)T)$ at the (k+1)st sampling point to the state $\underline{x}(kT)$ at the kth sampling point and to the control signal, $\underline{u}(kT)$. Fig. 1-1 is a block diagram representation of equation (1-1a). The blocks, in general, represent matrix multiplication as indicated by the double lined arrows.

Where convenient, the sampling period, T, will be dropped from the notation. It will be understood that the conversion from a specific sampling point to an instant in time will require the sampling period. Equation (1-1a) and (1-2a) will now be written

$$\underline{x}(k+1) = \phi(T)\underline{x}(k) + D(T)\underline{u}(k) \tag{1-1b}$$

$$y(k) = C\underline{x}(k) \tag{1-2b}$$

It should be noted that the physical quantities represented by the state variables and output variables are continuous functions of time. The values of the functions at the sampling instant are $\underline{x}(k)$, etc. For convenience and where no ambiguity results, the notations $\underline{x}(t)$, $\underline{x}(k)$, or simply $\underline{x}$ may be used interchangeably. This will hold also for command inputs, $\hat{r}(t)$ and $\underline{r}(t)$.

To optimally control a plant governed by equation (1-1a) it is desirable to express the control input $\underline{u}(k)$ as a function of the state and of all present and future desired states. The concept of "desired state" is a generalization of what is usually called "command input" or "desired output". If all of the state variables cannot be measured directly, and if the plant is "observable", it is possible to estimate the values of unmeasurable variables through use of the Kalman Filter [1,2]. It will be assumed that all of the state variables are available.

In general terms, the design of optimum control involves the determination of a control law or a control sequence so that the control process can be taken from a given initial state, $\underline{x}(0)$, to a desired state, $\underline{x}_d$, in some finite time. A "control law", $\underline{u}\left[\underline{x}(k),k\right]$, is the control signal expressed as a function of the state variables of the plant and possibly time. A "control sequence", $\underline{u}(k)$ $k=0,1,\ldots,$ is the control signal expressed as a function of discrete sampling instants.

1.3 The Performance Criterion

If a system is to be optimized, some criterion must be chosen that determines how well the system is operating. The criterion established depends upon the performance requirements of the system. The method used here is to assign a cost to the system at each sampling instant. The sum of these costs over the period $0 \leq t \leq NT$ is called the performance index and one form is given by

$$J_N = \sum_{k=1}^{N} \left[\underline{x}^T(k) Q \underline{x}(k) + \underline{u}^T(k-1) R \underline{u}(k-1) \right] \qquad (1\text{-}4)$$

where N is the number of sampling intervals to accomplish the desired state transition. The matrix Q and the matrix R are weighting factors which allow the relative cost of the errors and the control effort to be adjusted to achieve the desired performance. For optimal design, the performance index is to be minimized. The performance index will be discussed in detail in Chapter 3.

1.4 The Problem Statement

The problem can now be stated: Given the system defined by equation (1-1a), find the control sequence $\underline{u}(0), \underline{u}(1), \ldots, \underline{u}(N-1)$ such that for any initial state $\underline{x}(0)$ and some desired output, a performance index of the form given by equation (1-4) is minimized. Furthermore, determine if a stationary, linear feedback control law can be established such that the total cost over some period is optimum or "close" to optimum.

2. TIME OPTIMAL OR DEADBEAT RESPONSE

2.1 Problem Definition

It is desired that the system achieve equlibrium with zero steady-state error from any initial state as quickly as possible. In this and following sections, in the interest of simplicity, the desired state of the plant is identically zero for all t. It will also be assumed that the plant has a single input (m=1), i.e., u(k) is a scalar.

In order words, the object of control is to transfer any initial state $\underline{x}(0)$ to the equilibrium state $\underline{x}(N)=\underline{0}$ in a minimum number of sampling periods. It is desired to find the control sequence, u(0), u(1),...,u(N-1), and the control law which will accomplish this. A derivation of the control sequence is given from which a geometric interpretation can be obtained. The result is a stationary control law.

This problem may also be stated in terms of the performance index

$$J_N = \underline{x}^T(N)Q\underline{x}(N) \qquad (2\text{-}1)$$

where Q is the identity matrix and N indicates the final stage of the process. The problem then is to find the control sequence such that N is the smallest integer for which $J_N=0$ and $\underline{x}(t)=\underline{0}$ for all $t \geq NT$. Chapter 3 will show a dynamic programming solution of this problem.

2.2 Controllability

A plant is said to be completely controllable if there exists a control signal u(t), defined over a finite interval $t_o \leq t \leq t_1$, which moves the process from initial state $\underline{x}(t_o)$ to a desired equilibrium state $\underline{x}(t_1)$ in time interval $t_1 - t_o$. In the present case the state $\underline{x}(t_1)$ is the state $\underline{0}$.

A necessary and sufficient condition for complete controllability in the discrete-time case may be stated as follows: An nth order linear discrete-time process with a scalar control is completely controllable if, and only if, the vectors

$$
\begin{aligned}
\underline{s}_1 &= \phi(-T)\underline{d}(T) \\
\underline{s}_1 &= \phi(-2T)\underline{d}(T) \\
&\vdots \\
\underline{s}_n &= \phi(-nT)\underline{d}(T)
\end{aligned}
\tag{2-2}
$$

are linearly independent [1]. How these vectors arise will be shown.

Consider first a second-order system (n=2). Any two values of the state variables $x_1(k)$ and $x_2(k)$ define a point in the x_1, x_2 plane which is called the state space of the system. Analogously, any point in the state space is called a state. The evolution of the system from an arbitrary present state $(x_1(k), x_2(k))$ can be represented by a curve in the state space, called the trajectory, along which time is a parameter;

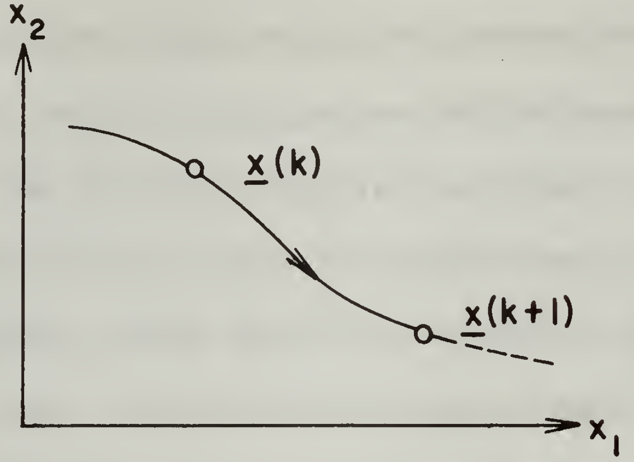

Fig. 2-1A TRAJECTORY IN STATE SPACE

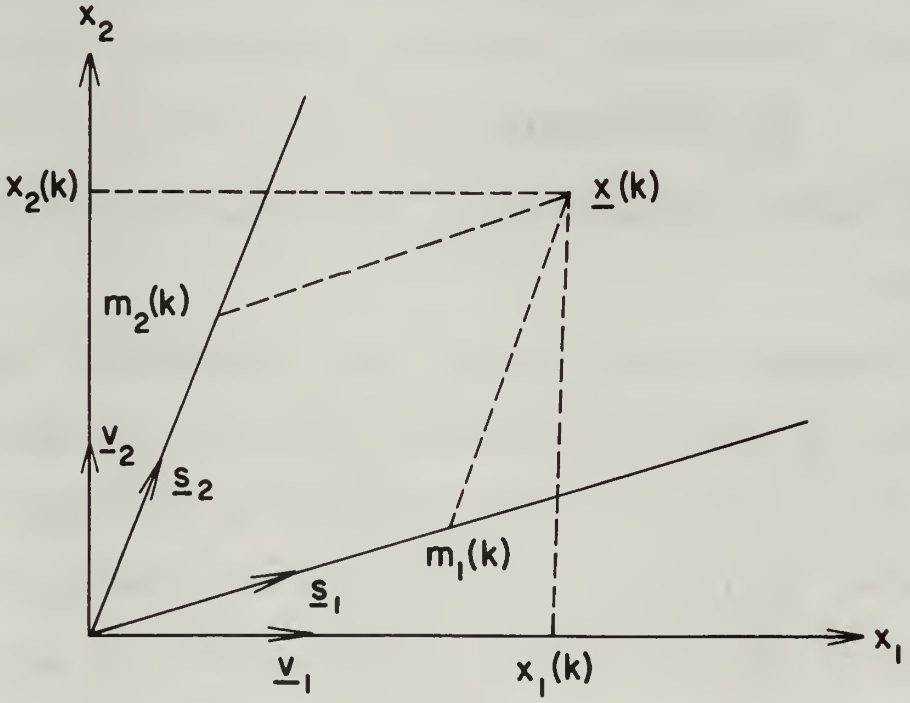

Fig. 2-2 THE CANONICAL AND ORTHOGONAL CO-ORDINATE SYSTEMS

see Fig. 2-1 where the arrow indicates the positive direction of time. Since the solution of a linear homogeneous differential equation, i.e., $u(t)=0$, is uniquely determined for all finite values of time by the initial state, there is one, and only one, trajectory passing through any given point in the x_1, x_2 plane. The behavior of the system may be thought of as the motion of a point along a trajectory. If the point does not move at all, the point is said to be an equilibrium state.

If $\underline{v}_1$, $\underline{v}_2$ are the unit vectors along the co-ordinate axes in the state space, then the state vector at any time kT can be written in the form (see Fig. 2-2)

$$\underline{x}(k) = x_1(k)\underline{v}_1 + x_2(k)\underline{v}_2 \tag{2-3}$$

Now, from what initial states $\underline{x}(0)$ is it possible to reach the origin of the state space in one sampling interval? The transition equation, again, describing the process is

$$\underline{x}(k+1) = \phi(T)\underline{x}(k) + \underline{d}(T)u(k) \tag{2-4}$$

For k=0, equation (2-4) yields

$$\underline{x}(1) = \phi(T)\underline{x}(0) + (T)u(0) \tag{2-5}$$

Thus, if it is assumed that $\underline{x}(1)=0$, the initial state from which this equilibrium condition can be reached in one sampling period is, solving equation (2-5) for $\underline{x}(0)$ and using equation (2-2),

$$\underline{x}(0) = -\phi(-T)\underline{d}(T)u(0) = -u(0)\underline{s}_1 \tag{2-6}$$

(A property of the transition matrix is that $\phi^k(T) = \phi(kT)$, e.g., $\phi^{-1}(T) = \phi(-T)$; see Appendix I). Equation (2-6) states that if u(0) is not bounded, all initial states $\underline{x}(0)$ lying along the vector $\underline{s}_1$ can be taken to $\underline{x}(T)=\underline{0}$ in one sampling period. The states satisfying equation (2-6) constitute a straight line in the state space.

The set of all states from which the origin can be reached in at most two steps is that set from which a state given by equation (2-6) can be reached in at most one step. For k=1, equation (2-4) yields

$$\underline{x}(0) = -u(0)\phi(-T)\underline{d}(T) - u(1)\phi(-2T)\underline{d}(T)$$

or

$$\underline{x}(0) = -u(0)\underline{s}_1 - u(1)\underline{s}_2 \tag{2-9}$$

If the vectors $\underline{s}_1$ and $\underline{s}_2$ are linearly independent, i.e., they do not lie along the same line through the origin of the state space, they form a basis in the state space. That is, any other vector in the state space can be written as a linear combination of $\underline{s}_1$ and $\underline{s}_2$.

It can be shown [1,5] that the vectors

$$\underline{s}_1 = \phi(-T)\underline{d}(T)$$

$$\underline{s}_2 = \phi(-2T)\underline{d}(T) \tag{2-10}$$

are linearly independent if, and only if

$$T\omega \neq q\pi \tag{2-11}$$

where q is any integer and ω is the imaginary part of any of the complex poles of the transfer function of the plant. This means that the input must excite all natural frequencies of the plant; in other words no cancellation of poles must be possible in the transfer function. In illustration consider a second-order process with the transfer function

$$G(s) = \frac{1}{s^2 + \omega^2}$$

From a flow graph of this process (see Chapter 3), or from the state equations

$$\dot{x}_1 = \omega x_2$$

$$\dot{x}_2 = -\omega x_1$$

and the methods in Appendix I, the following matrices result

$$\phi(T) = \begin{bmatrix} \cos\omega T & \sin\omega T \\ -\sin\omega T & \cos\omega T \end{bmatrix} \qquad \underline{d}(T) = \begin{bmatrix} -\cos\omega T/\omega \\ \sin\omega T/\omega \end{bmatrix}$$

Using equation (2-10)

$$\underline{s}_1 = \begin{bmatrix} -1/\omega \\ 0 \end{bmatrix} \qquad \underline{s}_2 = \begin{bmatrix} (-\cos\omega T \cos 2\omega T - \sin\omega T \sin 2\omega T)/\omega \\ (-\cos\omega T \sin 2\omega T + \sin\omega T \cos 2\omega T)/\omega \end{bmatrix}$$

By inspection it is seen that if $T\omega = q\pi$, where q is an integer, $\underline{s}_1 = \underline{+}\ \underline{s}_2$ and hence the two vectors are not linearly independent.

If the values of the control signal, u(0) and u(1), in equation (2-9) are not bounded, all initial states $\underline{x}(0)$ lying in the plane containing vectors $\underline{s}_1$ and $\underline{s}_2$ can be taken to the equilibrium state $\underline{x}(2) = \underline{0}$ in two sampling periods. If the values of the control signal are bounded, it would take a longer time to reach the equilibrium state.

The vectors $\underline{s}_1$ and $\underline{s}_2$ may be regarded as unit vectors along the axes of a new co-ordinate system, $\underline{x}(k)$ having the co-ordinates $m_1(k)$ and $m_2(k)$ in this new system, (see Fig. 2-2). This co-ordinate system will be called the canonical co-ordinate system.

2.3 Determination of the Control Sequence and Control Law

The next step is to determine the actual values of the sequence u(0), u(1), which will take an arbitrary initial state $\underline{x}(0)$ to the origin of the state space.

It was stated in the previous section that every initial state can be written in the form:

$$\underline{x}(0) = m_1(0)\underline{s}_1 + m_2(0)\underline{s}_2 \qquad (2\text{-}12)$$

where the m_i's are real numbers. If the process can be taken from the initial state $\underline{x}(0)$ to the equilibrium state in two sampling periods, it can be taken from the state $\underline{x}(1)$ to the equilibrium state in one sampling period. The state vector $\underline{x}(0)$ is a two-dimensional vector and the state vector $\underline{x}(1)$ is a one-dimensional vector in the canonical co-ordinate system. Then, $\underline{x}(1)$ may be expressed as

$$\underline{x}(1) = m_1(1)\underline{s}_1 \qquad (2\text{-}13)$$

Equation (2-4) with k=0 is

$$\underline{x}(1) = \phi(T)\ \underline{x}(0) + \underline{d}(T)u(0) \qquad (2\text{-}14)$$

Substituting equation (2-12) for $\underline{x}(0)$,

$$\underline{x}(1) = \phi(T)\left[m_1(0)\underline{s}_1 + m_2(0)\underline{s}_2\right] + \underline{d}(T)u(0) \qquad (2\text{-}15)$$

Simplifying using equation (2-10), the definitions of $\underline{s}_1$ and $\underline{s}_2$, and properties of the transition matrix, it follows that

$$\underline{x}(1) = m_1(0)\phi(T)\phi(-T)\underline{d}(T) + m_2(0)\phi(T)\phi(-2T)\underline{d}(T) + \underline{d}(T)u(0)$$

$$\underline{x}(1) = m_1(0)\underline{d}(T) + m_2(0)\underline{s}_1 + \underline{d}(T)u(0)$$

or

$$\underline{x}(1) = m_2(0)\underline{s}_1 + \left[m_1(0) + u(0)\right] \underline{d}(T) \tag{2-16}$$

Hence, $\underline{x}(1)$ will have the form required by equation (2-13) if, and only if,

$$u(0) = -m_1(0) \tag{2-17}$$

and

$$m_2(0) = m_1(1) \tag{2-18}$$

If $\underline{x}(1)$ is to be reduced to zero in one step, then

$$\underline{x}(2) = \underline{0} = \phi(T)\underline{x}(1) + \underline{d}(T)u(1) \tag{2-19}$$

Substituting equation (2-13) for $\underline{x}(1)$,

$$\underline{x}(2) = \phi(T)m_1(1)\underline{s}_1 + \underline{d}(T)u(1) \tag{2-20}$$

Using the definition $\underline{s}_1 = \phi(-T)\underline{d}(T)$,

$$\underline{x}(2) = \left[m_1(1) + u(1)\right]\underline{d}(T) \tag{2-21}$$

Therefore, the condition for $\underline{x}(1)$ to be reduced to zero in one step is

$$u(1) = -m_1(1) = -m_2(0) \tag{2-22}$$

which is the desired control signal at t=T. Hence, at every sampling instant, the optimal control signal u(k) is given by

$$u(k) = -m_1(k), \ (k=0,1 \text{ for a second-order process}) \tag{2-23}$$

where in general for any state,

$$\underline{x}(k) = m_1(k)\underline{s}_1 + m_2(k)\underline{s}_2, \; k=0,1 \tag{2-24}$$

The effect of the control will now be determined. Substituting equation (2-24) for $\underline{x}(k)$ into equation (2-4), and using the optimal control signal as given by equation (2-23),

$$\underline{x}(k+1) = \phi(T)\left[m_1(k)\underline{s}_1 + m_2(k)\underline{s}_2\right] - \underline{d}(T)m_1(k)$$

Simplifying, using equation (2-10), the definitions of $\underline{s}_1$ and $\underline{s}_2$,

$$\underline{x}(k+1) = m_1(k)\underline{d}(T) + m_2(k)\underline{s}_1 - \underline{d}(T)m_1(k)$$

$$\underline{x}(k+1) = m_2(k)\underline{s}_1 \tag{2-25}$$

Comparing this with equation (2-24), rewritten with k=k+1,

$$\underline{x}(k+1) = m_1(k+1)\underline{s}_1 + m_2(k+1)\underline{s}_2 \tag{2-26}$$

we see that after the application of the optimal control

$$m_1(k+1) = m_2(k) \tag{2-27}$$

In the general case for an nth order system, equations (2-24) and (2-25) are

$$\underline{x}(k) = m_1(k)\underline{s}_1 + m_2(k)\underline{s}_2 + \ldots + m_n(k)\underline{s}_n \tag{2-24a}$$

and

$$\underline{x}(k+1) = m_2(k)\underline{s}_1 + m_3(k)\underline{s}_2 + \ldots + m_n(k)\underline{s}_{n-1} \tag{2-25a}$$

and therefore

$$m_i(k+1) = m_{i+1}(k) \tag{2-27a}$$

A geometric interpretation of equations (2-23 to 2-27a) is as follows. The initial state of the plant may lie anywhere in the n-dimensional state space. After one sampling period and the application of a control signal as given by equation (2-23), the state can be expressed as a linear combination of only n-1 linearly independent vectors $\underline{s}_1, \ldots, \underline{s}_{n-1}$; in other words, after one sampling period, the initial state always lies in an (n-1)-dimensional space. The initial state is taken successively into a space of one less dimension during each sampling period. Ultimately, the state must be in a space of dimension zero which is an equilibrium point in the state space.

For the second-order case, the initial state $\underline{x}(0)$ lies anywhere in the $\underline{s}_1, \underline{s}_2$ plane. After the application of one control, the state $\underline{x}(1)$ lies on the line described by the vector $\underline{s}_1$. The final control takes the state to the origin.

2.4 Computing the Control Signal

According to equation (2-23), the optimal control signal is obtained by computing the first component in the canonical co-ordinate system, $m_1(k)$, of the state of the plant at each sampling instant. It will now be shown that the optimal control is given by

$$u(k) = \underline{a}^T \underline{x}(k) \qquad (2\text{-}28)$$

where $\underline{a}$ is an nx1 vector of constants, i.e., the control law is stationary.

Any component of a vector in one co-ordinate system can be expressed as a linear combination of the components of the same vector in a different co-ordinate system. Thus, it is always possible to write, considering the second-order process,

$$m_1(k) = a_1x_1(k) + a_2x_2(k) \tag{2-29}$$

where the a_i are real constants (see Fig. 2-3). It is then necessary to calculate these constants to obtain the control. Observe that by definition,

$$\underline{x}(k) = x_1(k)\underline{v}_1 + \underline{x}_2(k)\underline{v}_2 \tag{2-30}$$

and

$$\underline{x}(k) = m_1(k)\underline{s}_1 + m_2(k)\underline{s}_2 \tag{2-31}$$

In matrix notation, these expressions may be equated and written as

$$\left[\underline{v}_1 \vdots \underline{v}_2\right]\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \left[\underline{s}_1 \vdots \underline{s}_2\right]\begin{bmatrix} m_1(k) \\ m_2(k) \end{bmatrix} \tag{2-32}$$

By definition

$$\begin{aligned}\left[\underline{s}_1 \vdots \underline{s}_2\right] &= \left[\phi(-T)\underline{d}(T) \vdots \phi(-2T)\underline{d}(T)\right] \\ &= \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = [P]\end{aligned} \tag{2-33}$$

The basis vectors $\underline{s}_1, \underline{s}_2$ are related to the basis vectors $\underline{v}_1, \underline{v}_2$ by

$$\left[\underline{s}_1 \vdots \underline{s}_2\right] = \left[\underline{v}_1 \vdots \underline{v}_2\right]\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \tag{2-34}$$

This equation states that

$$\begin{aligned}\underline{s}_1 &= p_{11}\underline{v}_1 + p_{21}\underline{v}_2 \\ \underline{s}_2 &= p_{12}\underline{v}_1 + p_{22}\underline{v}_2 \quad \text{(see Fig. 2-4)}\end{aligned} \tag{2-35}$$

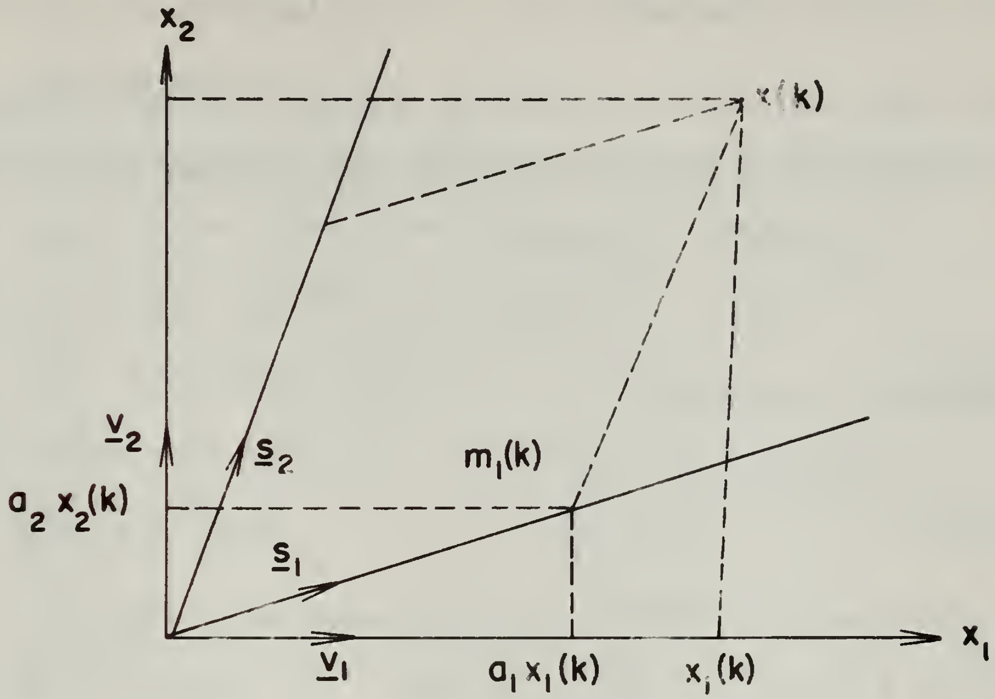

Fig. 2-3 VECTOR COMPONENTS

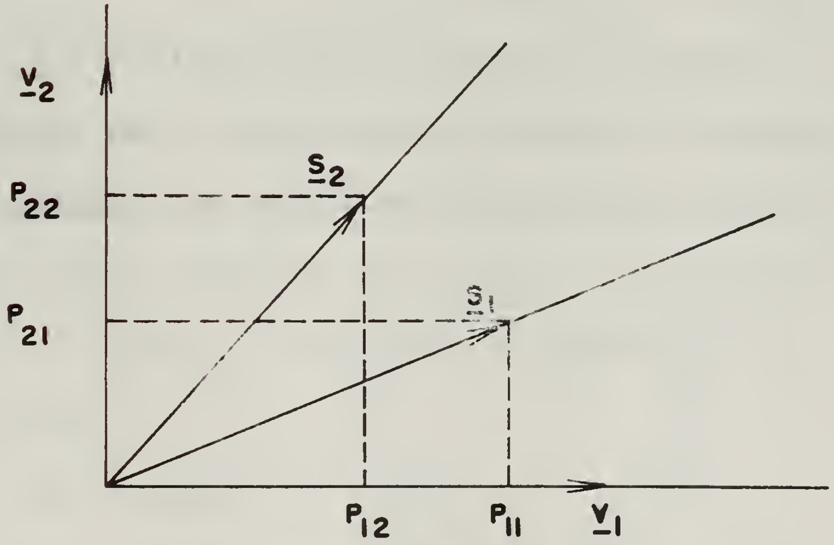

Fig. 2-4 COMPONENTS OF THE CANONICAL BASIS VECTORS IN THE ORTHOGONAL CO-ORDINATE SYSTEM

Substituting equation (2-34) in equation (2-32),

$$\left[\underline{v}_1 \vdots \underline{v}_2\right] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \left[\underline{v}_1 \vdots \underline{v}_2\right] \left[P\right] \begin{bmatrix} m_1(k) \\ m_2(k) \end{bmatrix} \qquad (2\text{-}36)$$

Note that the vectors $\underline{v}_1$ and $\underline{v}_2$ are unit vectors in an orthogonal co-ordinate system, and hence, have the values

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} ; \quad \underline{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Therefore the matrix

$$\left[\underline{v}_1 \vdots \underline{v}_2\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is the identity matrix. Solving equation (2-36),

$$\begin{bmatrix} m_1(k) \\ m_2(k) \end{bmatrix} = \left[P\right]^{-1} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \qquad (2\text{-}37)$$

or in general for an nth-order system,

$$\underline{m}(k) = P^{-1}\underline{x}(k) \qquad (2\text{-}38)$$

where P^{-1} denotes the inverse of the matrix $P = \left[\underline{s}_1 \vdots \underline{s}_2 \vdots \quad \vdots \underline{s}_n\right]$. If the system is completely controllable, the required inverse exists. Thus, equations (2-23) and (2-38) together define the control law $u\left[\underline{x}(k)\right]$: Letting

$$P^{-1} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}, \qquad (2\text{-}39)$$

$$u(k) = -m_1(k) = -\underline{a}^T\underline{x}(k) \qquad (2\text{-}40)$$

where $\underline{a}^T$ is the notation for a vector transpose and is the first row of the matrix P^{-1}. The optimum control law is therefore constant feedback of the state variables at sampling instants. Fig. 2-5 shows the block diagram for an arbitrary second-order process implementing the stationary feedback law when the desired state is $\underline{0}$.

The control sequence can also be found as a function of the state $\underline{x}(0)$. From equations (2-23) and (2-27a), the following relations may be derived

$$u(k) = -m_1(k) = -m_{k+1}(0) \tag{2-41}$$

Equation (2-38) with k=0 is

$$\underline{m}(0) = P^{-1}\underline{x}(0) \tag{2-42}$$

Hence,

$$\underline{u} = -P^{-1}\underline{x}(0) \tag{2-43}$$

where

$$\underline{u} = \begin{bmatrix} u(0) \\ u(1) \\ \cdot \\ \cdot \\ \cdot \\ u(n-1) \end{bmatrix} \tag{2-44}$$

The optimal control was derived here for a second-order system but holds for an nth order system where the initial state vector can be expressed as

$$\underline{x}(0) = -u(0)\underline{s}_1 - u(1)\underline{s}_2 - \ldots - u(n-1)\underline{s}_n \tag{2-45}$$

If the n vectors

$$\underline{s}_i = \phi(-iT)\underline{d}(T), \; i = 1,2,\ldots,n \tag{2-46}$$

are linearly independent and if the values u(i), i=0,1,..., n-1, are not bounded, the initial state $\underline{x}(0)$ can be brought to

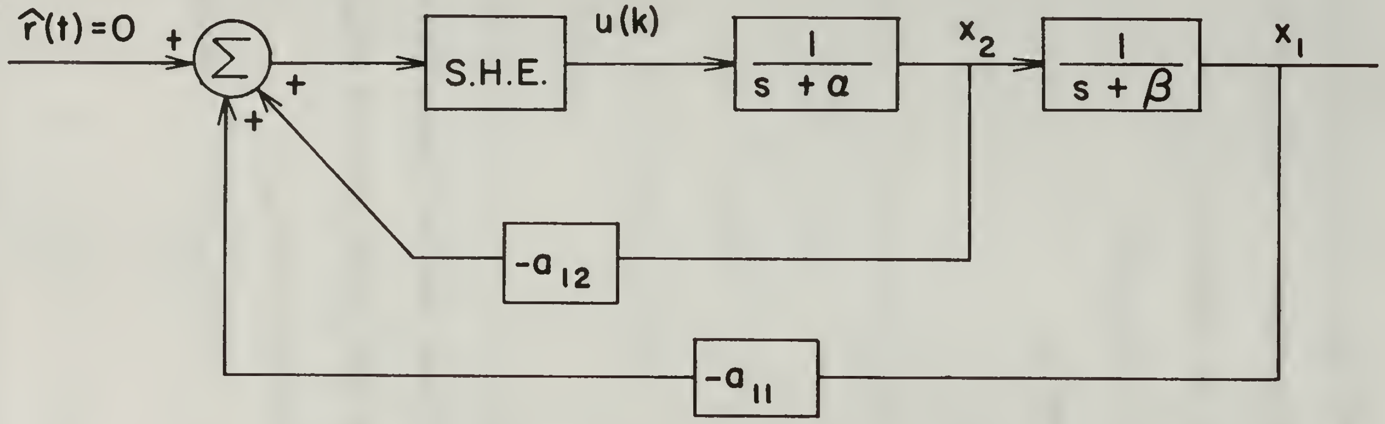

Fig. 2-5 OPTIMAL REGULATOR SYSTEM

the equilibrium state $\underline{x}(N)=\underline{0}$ in at most n samples; that is, $t_1 \leq nt$. The control law was shown to be stationary. For a derivation with vector inputs see [4].

2.5 Arbitrary Polynomial Inputs [5]

The foregoing discussion shows the derivation of the time-optimal control law when the desired equilibrium point with zero error is the origin of the state space. The solution will now be extended for a follow-up system. This problem is equivalent to making the first component, $y_1(k)$, of the plant output vector

$$\underline{y}(k) = C\underline{x}(k) \tag{1-2a}$$

equal to some desired output, $\hat{r}(t)$ or $\hat{r}(k)$, in such a way that the plant is in equilibrium with zero system error.

If $\hat{r}(t)$ is an arbitrary function of time, the problem cannot be solved in general. However, when $\hat{r}(t)$ has the polynomial form

$$\begin{aligned} \hat{r}(t) &= g_1 + g_2 t + \dots + g_p t^{p-1} \ ; \ t \geq 0 \\ &= 0 \ ; \ t < 0 \end{aligned} \tag{2-47}$$

(where the g_i are arbitrary real numbers) the solution can be obtained by modifying the solution of the previous regulator problem.

Note that the components of the plant output vector, $\underline{y}(k)$, are the physical variables of the plant which can be measured directly. These components are linear combinations of the plant state variables. For example y_1 may be the sum of the state variables x_1 and x_2. It will be assumed at this point, for simplicity, that

$$y_1 = x_1 \tag{2-48}$$

It is known that the optimal control requires knowledge of all of the plant state variables. It then seems feasible to derive from $\hat{r}(t)$ a command input state vector, $\underline{r}(t)$, to which it is desired to drive the plant state vector so that the plant is in equilibrium with zero system error. Then, from equation (2-47), form

$$\begin{aligned} r_1(t) &= \hat{r}(t) = g_1 + g_2 t + \ldots + g_p t^{p-1} \\ r_2(t) &= \frac{d\hat{r}(t)}{dt} = g_2 + 2g_3 t + \ldots + (p-1) g_p t^{p-2} \\ &\ldots\ldots\ldots\ldots \\ r_p(t) &= \frac{d^{(p-1)}\hat{r}(t)}{dt^{(p-1)}} = (p-1)\ (p-2) \ldots (p-(p-1)) g_p \end{aligned} \tag{2-49}$$

Also, note that

$$\frac{d^p\hat{r}(t)}{dt^p} = 0 \tag{2-50}$$

Then

$$\underline{r}(t) = \begin{bmatrix} r_1(t) \\ \cdot \\ \cdot \\ \cdot \\ r_p(t) \end{bmatrix} \tag{2-51}$$

Note that equations (2-49) require knowledge of the derivatives of the input. This may be difficult to obtain in some cases.

If the plant contains at least p "free" integrators (without feedback), that is, the plant has a type number p, then the differential equations of the plant in terms of the state variables are

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_{\cancel{2}\,3}$$

$$\cdots\cdots\cdots$$

$$\dot{x}_p = x_{p+1} \qquad (2\text{-}52)$$

plus other equations which do not involve the state variables $x_1,\ldots,x_p$. Form the modified state variables

$$\hat{x}_1 = x_1 - r_1$$

$$\hat{x}_2 = x_2 - r_2$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\hat{x}_p = x_p - r_p$$

$$\hat{x}_i = x_i \ , \ p < i \leq n \qquad (2\text{-}53)$$

Substituting equation (2-53) into equation (2-52) shows that the modified state variables obey the same differential equations. Hence, in equilibrium when $\hat{\underline{x}}=0$,

$$x_1 = r_1$$

$$x_2 = r_2$$

$$\cdots\cdots$$

$$x_p = r_p$$

$$x_i = 0 \ , \ p < i \leq n \qquad (2\text{-}54)$$

which shows that the output will be identical with the input after, at most, n sampling periods.

Thus, the regulator system which contains at least p free integrators is converted to a follow-up system simply by taking the difference between the plant state $\underline{x}(k)$ and the input state $\underline{r}(k)$ to create the modified state $\hat{\underline{x}}$ prior to multiplying by the feedback coefficients a_{1j}. In matrix notation, then,

$$u(k) = -\underline{a}^T\left[\underline{x}(k) - \underline{r}(k)\right] \qquad (2\text{-}55)$$

(the components, $r_{p+1}, \ldots, r_n$, of vector $\underline{r}$ will be zero).

Fig. 2-6 shows the block diagram for a simple second-order plant with one free integrator and, therefore, the input $r(t)$ = constant ($p=1$). Fig. 2-7 shows the block diagram of a follow-up system for the general case where $u(k)$ is given by equation (2-55).

The foregoing solution states that the plant must have at least p free integrators, that is, it must be at least of type number p in order to follow an input $\hat{r}(t)$ that satisfies the differential equation

$$\frac{d^p\hat{r}(t)}{dt^p} = 0 \qquad (2\text{-}50)$$

For example, a plant of type number 2 will follow only "step" and "ramp" inputs ($p \leq 2$) with zero steady-state error. However, this restriction on the allowable order of the input to a system with a given number of free integrators may be relaxed somewhat as follows.

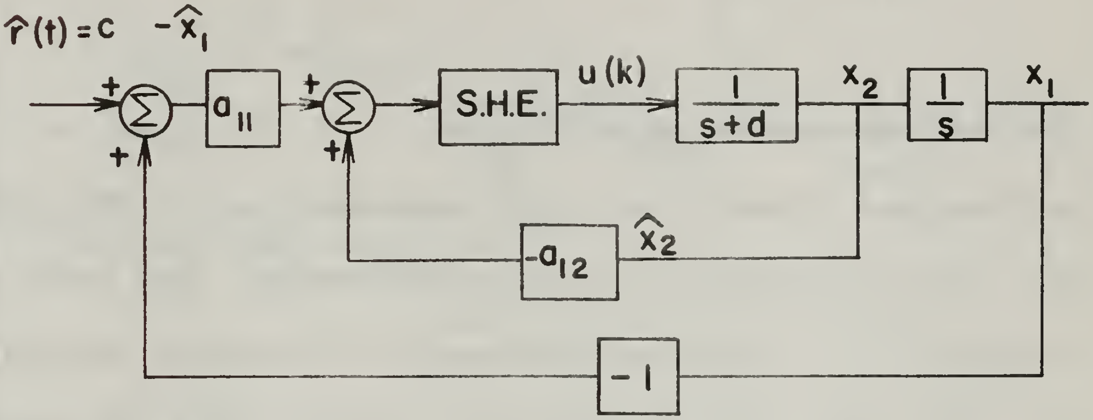

Fig. 2-6 FOLLOW-UP SYSTEM EXAMPLE

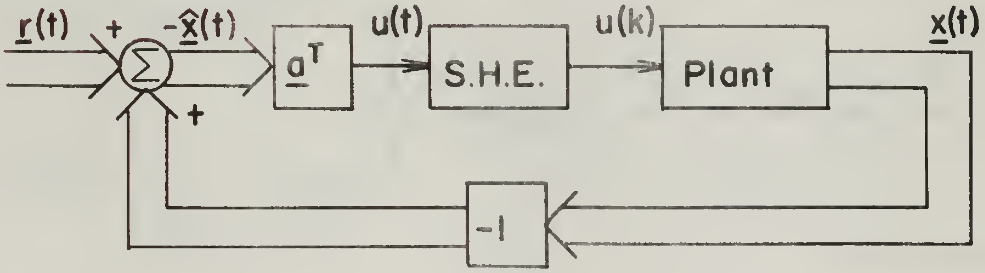

Fig. 2-7 VECTOR BLOCK DIAGRAM OF FOLLOW-UP SYSTEM

Consider the example of Fig. 2-6 but where the command input is a "unit ramp",

$$\hat{r}(t) = t$$

or, using equations (2-49)

$$\underline{r}(t) = \begin{bmatrix} t \\ 1 \end{bmatrix} \qquad (2\text{-}56)$$

The desired response then is $x_1(\infty) = t$, and $x_2(\infty) = 1$. Looking at a signal-flow graph representation of the system, Fig. 2-8 (disregarding R), we note, however, that there is a feedback of x_2 upon itself through an integrator and gain of -d. This means that x_2 could not possibly hold the steady-state value of 1. Stated another way

$$\dot{x}_2(\infty) = u(\infty) - d \cdot x_2(\infty)$$
$$0 \neq 0 - d \cdot (1) \qquad (2\text{-}57)$$

We can get around this difficulty by introducing an auxiliary input, R, as shown in Fig. 2-8. Equation (2-57) could then be written

$$\dot{x}_2(\infty) = R = d \cdot x_2(\infty)$$
$$0 = R - d \cdot (1) \qquad (2\text{-}58)$$

Therefore,

$$R = d \cdot (1) \qquad (2\text{-}59)$$

It is seen, then, that through the use of an auxiliary input a plant can be made to follow a polynomial input of one order higher than the type number of the plant. The introduction of

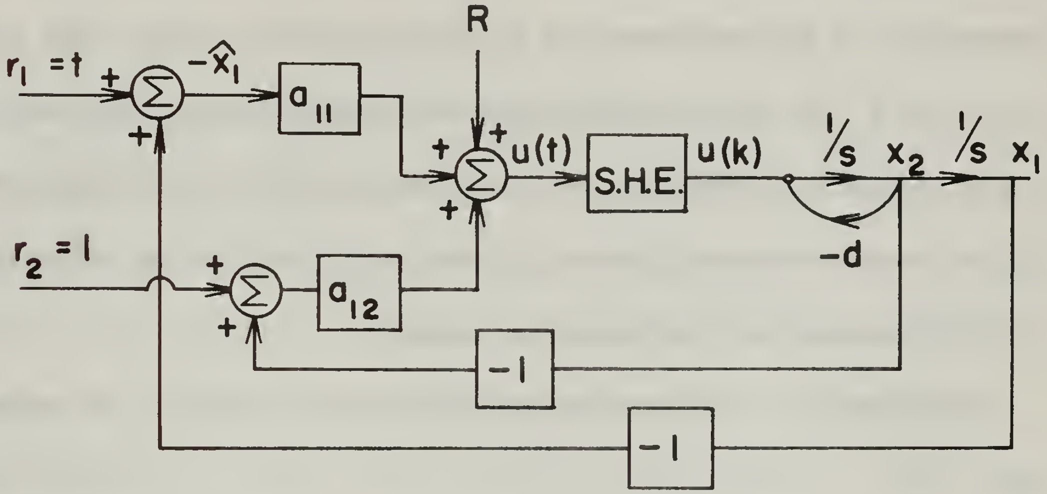

Fig. 2-8 FOLLOW-UP SYSTEM EXAMPLE
WITH AUXILIARY INPUT

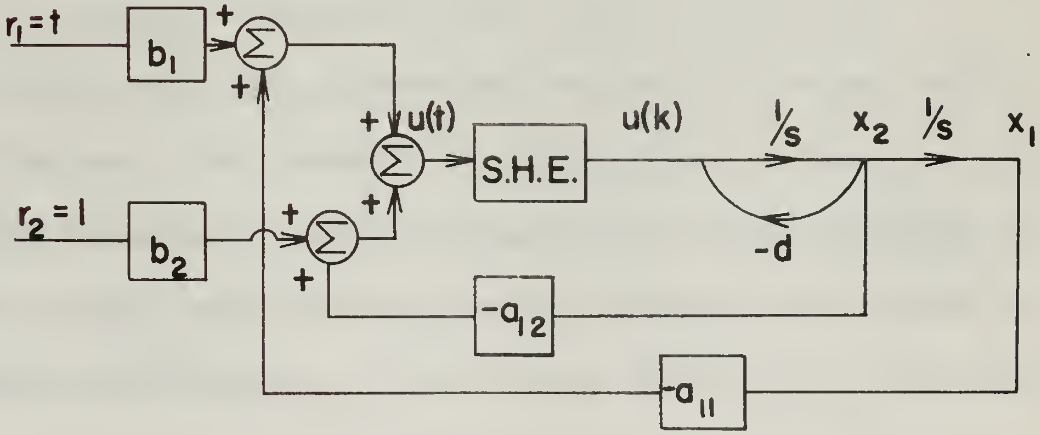

Fig. 2-9 FOLLOW-UP SYSTEM EXAMPLE
FOR ARBITRARY (IN A CLASS) INPUT

the auxiliary input simply results in a shift in the origin of the canonical co-ordinate system. The equilibrium state is still reached in $k \leq n$ stages.

It is seen from equation (2-59) that, if the auxiliary input is to be implemented as shown in Fig. 2-8, the command input and its derivatives must be known beforehand and may not be arbitrary. However, by implementing R as a function of the input state vector $\underline{r}$, the system will follow an arbitrary, but still restricted, polynomial input.

Continuing with the example of Fig. 2-8, it is seen that

$$u = - a_{11}(x_1 - r_1) - a_{12}(x_2 - r_2) + R \qquad (2\text{-}60)$$

where

$$R = d \cdot r_2 \qquad (2\text{-}61)$$

Then substituting for R, and rearranging,

$$u = a_{11}r_1 + (a_{12} + d)\, r_2 - a_{11}x_1 - a_{12}x_2 \qquad (2\text{-}62)$$

If the system is redefined as shown in Fig. 2-9, and if $b_1 = a_{11}$ and $b_2 = (a_{12} + d)$, it is seen that u is still given by equation (2-62). The input vector, $\underline{r}$, may now represent an arbitrary polynomial, in a class, and the origin of the canonical co-ordinate system will be automatically shifted so that the equilibrium state may be reached.

In general matrix notation the control, u(k), is given by

$$u(k) = \underline{b}^T\underline{r}(k) - \underline{a}^T\underline{x}(k) \qquad (2\text{-}63)$$

where

$$\underline{b}^T = \left[b_1 \; b_2 \; \ldots b_p \; b_{p+1} \; \ldots b_n\right] \tag{2-64}$$

(the elements $b_{p+1} \ldots b_n$, where $p < n$, are zero and must be included in the vectors $\underline{b}$ and $\underline{r}$ so that they will be conformable with the (nxl) vectors $\underline{a}$ and $\underline{x}$.) Elaboration is required for the general formulation of the vector $\underline{b}$.

It can be shown [6] that for a scalar control, u(t), and a completely controllable system, the coefficient matrix A of equation (1-1) can be written in a canonical form. This results in the signal-flow graph representation of any plant as shown generally in Fig. 2-10 (output not shown). The plant shown has m free integrators. From the preceding example it can be seen that the elements of the vector $\underline{b}$ are formed as follows:

$$b_i = a_{1i} + 0 \; ; \; i = 1, \ldots, m$$

$$b_p = a_{1p} + 0 \; ; \; p < m+1$$

$$b_p = a_{1p} + d_{m+1} \; ; \; p = m+1$$

$$b_j = 0 \qquad ; \; j = p+1, \ldots, n; \; p < n$$

where p is the order of the polynomial input and where d_{m+1} is shown in Fig. 2-10. The a_{1i} are the optimal gains given by the first row of P^{-1}.

It has been shown that a plant can be made to follow a polynomial input function of order one higher than the number of free integrators in the plant. This will be illustrated in Chapter 4.

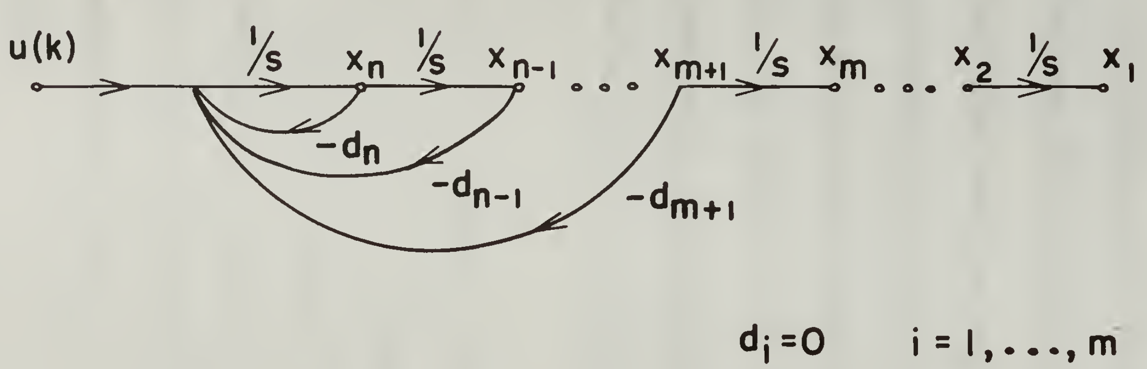

Fig. 2-10 SIGNAL-FLOW GRAPH OF A PLANT IN CANONICAL FORM

The state of the plant, $\underline{x}(t)$, is a continuous function of time and is being fed back continuously, while the control input is constant over the sampling period. Thus, if the desired value of any of the state variables is not a constant, and if it is being fed back to the input, then the error at the input will not be zero at all times and therefore the system will not be in equilibrium. This will also be illustrated in Chapter 4.

If the plant output is other than that given by equation (2-48), $y_1 = x_1$, the command input vector $\underline{r}$ must be written in a form to correspond with the desired values of the state variables. For example if the output of the second-order plant shown in Fig. 2-9 is given by

$$y_1 = x_1 + x_2,$$

and the desired output is a ramp,

$$\hat{r}(t) = gt$$

then for zero error

$$y_1 = gt = x_1 + x_2 \tag{2-66}$$

Taking the derivative,

$$g = \dot{x}_1 + \dot{x}_2$$

or

$$g = x_2 + 0 \; ; \; \dot{x}_1 = x_2 \tag{2-67}$$

for equilibrium. Solving equations (2-66) and (2-67) simultaneously,

$$x_1 = g(t-1) = r_1$$

$$x_2 = g = r_2 \tag{2-68}$$

Equation (2-68) gives the desired values of the state variables and, hence, the command input vector.

3. GENERALIZED PERFORMANCE INDICES AND DYNAMIC PROGRAMMING

3.1 General

In this chapter dynamic programming will be applied to optimize the performance of a dynamic system. The results are applicable to both sampling and continuous systems, a fact which will be discussed later in the chapter. This approach leads to a stationary control law for the infinite stage regulator problem.

3.2 Problem Definition and Development

Consider the system discussed in Chapter 1 described by the difference equation

$$\underline{x}(k+1) = \phi_x(T)\,\underline{x}(k) + D_x(T)\underline{u}(k) \tag{3-1}$$

A performance criterion is established to measure "how well" the system's output follows the command input $\hat{r}(t)$. This is called the performance index and is denoted by J. Fig. 3-1 shows schematically how the performance index is evaluated.

Inputs: The method of dynamic programming requires the knowledge of all the desired states; therefore, the system will be optimized for an expected class of command inputs $\left\{\hat{r}(t)\right\}$. A member $\hat{r}(t)$ of this class of inputs is defined by

$$\begin{aligned} \hat{r}(t) &= r_1(t) \quad ; \; t \geq 0 \\ &= 0 \qquad ; \; t < 0 \end{aligned} \tag{3-2}$$

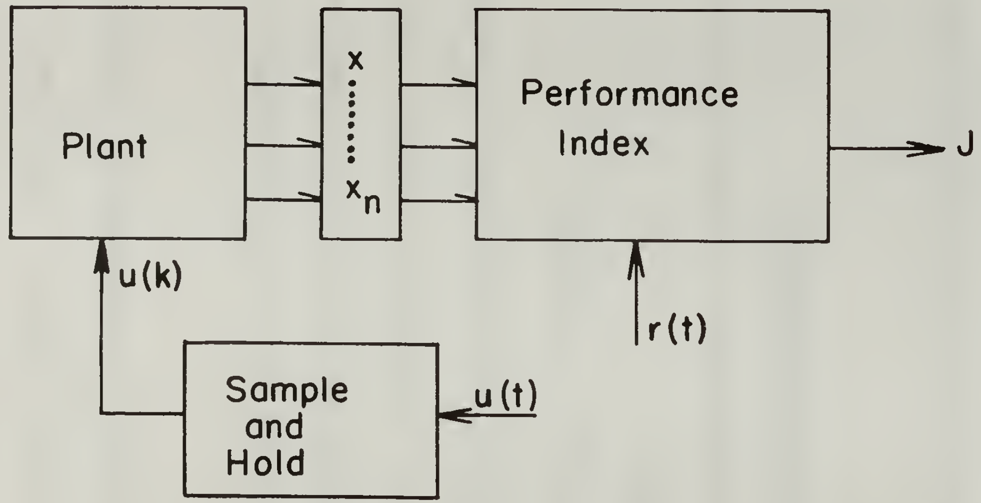

Fig. 3-1 EVALUATION OF PERFORMANCE INDEX

where

$$\underline{r}(t) = \phi_r(t)\underline{w}(0) = \begin{bmatrix} r_1(t) \\ \cdot \\ \cdot \\ r_p(t) \end{bmatrix} \qquad (3\text{-}3)$$

$\phi_r(t)$ is the transition matrix of a pth order linear differential equation with constant coefficients and $\underline{w}(0)$ is an arbitrary vector. ϕ_r determines the class of inputs for which the system is to be optimized and $\underline{w}(0)$ determines a particular member of that class.

Example 1: To illustrate the formulation of ϕ_r, suppose a member of the class of inputs $\left\{\hat{r}(t)\right\}$ satisfies the differential equation

$$\frac{d^2\hat{r}(t)}{dt^2} = 0 \qquad (3\text{-}4)$$

The equations for the state variables, (2-49), lead to the flowgraph shown in Fig. 3-2. ϕ_r may be determined by inspection of the flowgraph:

$$\phi_r(t) = \mathcal{L}^{-1}\begin{bmatrix} 1/s & 1/s^2 \\ 0 & 1/s \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \qquad (3\text{-}5)$$

Then, from equation (3-3),

$$r_1(t) = w_1(0) + tw_2(0) \qquad (3\text{-}6)$$

Example 2: The solution to the harmonic equation

$$\ddot{w}_1 + \omega^2 w_2 = 0 \; ; \; t \geq 0 \qquad (3\text{-}7)$$

yields a sinusoidal signal. This equation may be decomposed into

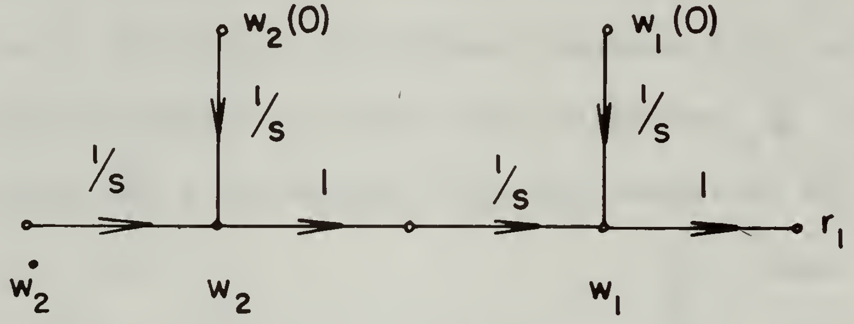

Fig. 3-2 FLOWGRAPH FOR POLYNOMIAL EXAMPLE

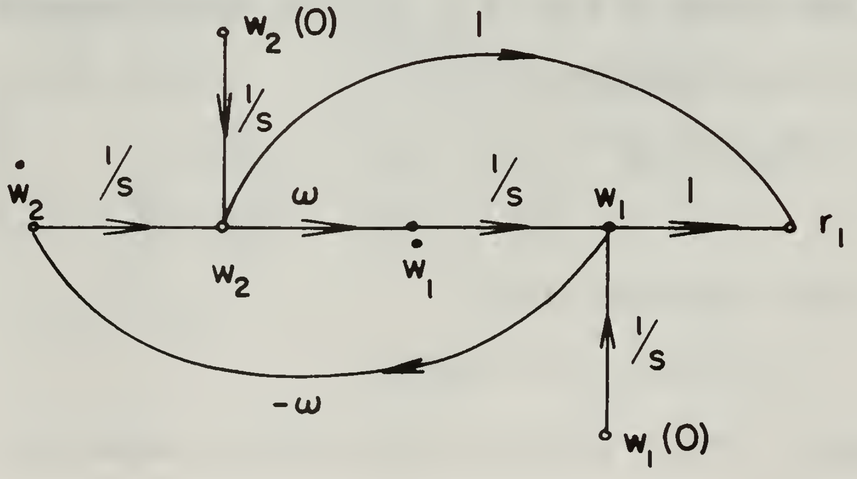

Fig. 3-3 FLOWGRAPH FOR SINUSOIDAL EXAMPLE

$$\dot{w}_1 = \omega w_2$$

$$\dot{w}_2 = -\omega w_1 \qquad (3\text{-}8)$$

From the state equations or from the flowgraph shown in Fig. 3-3, ϕ_r is seen to be

$$\phi_r(t) = \mathcal{L}^{-1} \begin{bmatrix} \frac{s}{s^2+\omega^2} & \frac{\omega}{s^2+\omega^2} \\ \frac{-\omega}{s^2+\omega^2} & \frac{s}{s^2+\omega^2} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix} \qquad (3\text{-}9)$$

Then

$$r_1(t) = w_1(0)\cos\omega t + w_2(0)\sin\omega t \qquad (3\text{-}10)$$

If a system has been designed for optimal performance for a particular class of inputs, it will approach nearly optimal performance with any other input signal which can be closely approximated by an r(t) over one sampling interval [3].

Performance Index: The error between command input and plant output is given by

$$e(t) = \hat{r}(t) - y_1(t) \qquad (3\text{-}11)$$

The undesirability of error is not usually a function of the sign, so a more plausible measure would be

$$e(t) = |\hat{r}(t) - y_1(t)|$$

or

$$e(t) = (\hat{r}(t) - y_1(t))^2 \qquad (3\text{-}12)$$

The second is commonly used because of its mathematical convenience. A generalized measure of error is defined as the quadratic form

$$\underline{z}^T(t) Q \underline{z}(t) \geq 0 \tag{3-13}$$

where Q is an (n+p) by (n+p) positive semi-definite matrix, and $\underline{z}(t)$ is an (n+p) by 1 vector defined by

$$\underline{z}(t) = \left[\begin{array}{c} \underline{x}(t) \\ \hline \underline{r}(t) \end{array}\right] \tag{3-14}$$

Example: To illustrate the preceding development, suppose the desired measure of error is

$$(r_1(t) - x_1(t))^2$$

Then, the Q matrix is given by

$$q_{1,1} = q_{n+1,n+1} = 1$$

$$q_{1,n+1} = q_{n+1,1} = -1$$

$$\text{all other elements} = 0 \tag{3-15}$$

To complete the notation in terms of the combined state vector $\underline{z}(t)$, define

$$D(T) = \left[\begin{array}{c} D_x(T) \\ \hline D_r(T) \end{array}\right] \tag{3-16}$$

$$\phi(T) = \left[\begin{array}{c|c} \phi_x(T) & 0 \\ \hline 0 & \phi_r(T) \end{array}\right] \tag{3-17}$$

where $D_x(T)$, $\phi_x(T)$ and $\phi_r(T)$ are as previously defined, and $D_r(T)$ is $\underline{0}$. The latter follows from the definition of $\underline{r}(t)$ in equation (3-3) where there is no $D_r(T)$ term.

The two state transition difference equations are combined as

$$\underline{z}(k+1) = \phi(T)\underline{z}(k) + D(T)\underline{u}(k) \qquad (3\text{-}18)$$

In many systems the control energy represents cost and, therefore, it may be desired to consider control energy in the measure of performance as well as the system's errors. The quadratic form is also used for measuring control effort

$$\underline{u}^T(t)R\underline{u}(t)$$

As stated previously, if the performance of a system is to be optimized, a criterion must be selected to tell "how well" it is operating. This criterion is the performance index, which, over the period $0 \leq t \leq NT$ is given by

$$J_N\left[\underline{z}(0),\underline{u}(T),\ldots,\underline{u}((N-1)T)\right]$$

$$= \int_0^{NT} \left[\underline{z}^T(t)Q\underline{z}(t) + \underline{u}^{*T}(t)R\underline{u}^*(t)\lambda(t)dt\right] \qquad (3\text{-}19)$$

where Q and R are proportional weighting matrices which compromise between minimization of errors and control energy. $\lambda(t)$ is a positive time-variant weighting function. Note that the smaller J_N the more satisfactory the system's performance.

The foregoing definition is very general and by varying Q,R and $\lambda(t)$ the index is adjusted to fit problems of physical significance.

Example 1: The response of a plant is never instantaneous, and as a result for some period of time following the application of the control signal, errors are inevitable. As time progresses, these errors are expected to become smaller. Therefore, it may be desirable to have $\lambda(t)$ increase with time. If the errors and the control effort are to be measured only at the sampling instants and given zero weight at all other times, then,

$$J_N = \sum_{k=1}^{N} \left[\lambda(k) \; (\underline{z}^T(k)Q\underline{z}(k) + \underline{u}^T(k-1)R\underline{u}(k-1))\right] \tag{3-20}$$

where $\lambda(k)$ is defined as the value of $\lambda(t)$ at the kth sampling instant. If the weighting of errors and control effort remains the same at all sampling instants, then $\lambda(k)$ is dropped from the performance index.

Example 2: A performance index concerned only with the error at the Nth sampling instant is

$$J_N = \underline{z}^T(N)Q\underline{z}(N) \tag{3-21}$$

If Q is the identity matrix, this index is the Euclidean norm of the system's state variables at the end of the Nth, or final stage of the process. Note that it reflects no concern whatever for what goes on in the early stages of the process, nor for the control effort required. It reflects the desire to bring the system as close as possible to the origin.

If a system has deadbeat response as described in Chapter 2, it is possible to select a control signal that will make $J_N = 0$ for a finite N and any initial state $\underline{z}(0)$.

3.3 Recurrence Relations [7]

It is now possible to state the problem more precisely. Given a linear plant defined by equation (3-1) and a class of input signals defined by equations (3-2) and (3-3), find the sampled control signal $\underline{u}(k)$ for the $t \geq 0$ which minimizes the performance index. That is, find the sequence of vectors $\underline{u}(0)$, $\underline{u}(1)$, $\underline{u}(2)$,... such that for any initial state $\underline{z}(0)$, the performance index J_N, equation (3-19), is minimized. The performance index considered is

$$J\left[\underline{z}(0)\right] = \sum_{k=1}^{N} \left[\underline{z}^T(k)Q\underline{z}(k) + \underline{u}^T(k-1)R\underline{u}(k-1)\right] \qquad (3\text{-}22)$$

which is the case where $\lambda(k)$ is dropped from equation (3-20).

The optimum return function $J_N^o\left[\underline{z}(0)\right]$ is defined as the minimum value of $J_N\left[\underline{z}(0)\right]$ and has the known simple form

$$J_N^o\left[\underline{z}(0)\right] = \underline{z}^T(0)P_N\underline{z}(0) \qquad (3\text{-}23)$$

which is proved by induction later. P_N is a symmetric, positive semi-definite matrix. By definition

$$J_{N+1}\left[\underline{z}(0)\right] = \min_{\underline{u}(0)} \min_{\underline{u}(1)} \cdots \min_{\underline{u}(N)} \sum_{k=1}^{N+1} \left[\underline{z}^T(k)Q\underline{z}(k) + \underline{u}^T(k-1)R\underline{u}(k-1)\right] \qquad (3\text{-}24)$$

$\underline{z}(1)$ is determined solely by $\underline{u}(0)$, which enables equation (3-24) to be written as

$$J_{N+1}\left[\underline{z}(0)\right] = \min_{\underline{u}(0)}\left\{ \underline{z}^T(1)Q\underline{z}(1) + \underline{u}^T(0)R\underline{u}(0) + \min_{\underline{u}(1)}, \min_{\underline{u}(2)},\ldots,\min_{\underline{u}(N)} \sum_{k=2}^{N+1} \left[\underline{z}^T(k)Q\underline{z}(k) + \underline{u}^T(k-1)R\underline{u}(k-1)\right]\right\} \tag{3-25}$$

The last group of terms is exactly $J_N^o\left[\underline{z}(1)\right]$ which gives

$$J^o_{N+1}\left[\underline{z}(0)\right] = \min_{\underline{u}(0)}\left\{ \underline{z}^T(1)Q\underline{z}(1) + \underline{u}^T(0)R\underline{u}(0) + J_N^o\left[\underline{z}(1)\right]\right\} \tag{3-26}$$

This equation can be arrived at directly using the principle of optimality, which states that the minimum cost of an N+1 stage process is the minimum of the sum of the cost of the first stage and the minimum cost of the N remaining stages. Note that the arguments of the state and control variables increase with time, whereas the subscript on J_N decreases with time.

Substituting equations (3-18) and (3-23) into (3-26)

$$J_{N+1}\left[\underline{z}(0)\right] =$$

$$= \min_{\underline{u}(0)}\left\{\left[\phi\underline{z}(0) + D\underline{u}(0)\right]^T(Q + P_N)\left[\phi\underline{z}(0) + D\underline{u}(0)\right] + \underline{u}^T(0)R\underline{u}(0)\right\} \tag{3-27}$$

Completing the square on the right side of (3-27) and defining

$$\underline{u}'(0) = A_{N+1}\underline{z}(0) \tag{3-28}$$

$$A_{N+1} = -\left[D^T(Q + P_N)D + R\right]^{-1}D^T(Q + P_N)\phi \tag{3-29}$$

Equation (3-27) becomes

$$J^{o}_{N+1}\left[\underline{z}(0)\right] = \min_{\underline{u}(0)} \Big\{ \left[\underline{u}(0) - \underline{u}'(0)\right]^{T} \left[D^{T}(Q + P_{N})D + R\right]\left[\underline{u}(0) - \underline{u}'(0)\right]$$

$$+ \underline{z}^{T}(0)\phi^{T}(Q + P_{N})\phi\underline{z}(0)$$

$$- \underline{z}^{T}(0)A_{N+1}\left[D^{T}(Q + P_{N})D + R\right]A_{N+1}\,\underline{z}(0)\Big\} \qquad (3\text{-}30)$$

The control $\underline{u}(0)$ occurs only in the first term. If the matrix $\left[D^{T}(Q + P_{N})D + R\right]$ is positive definite the optimal control is unique. Then, the minimum value of the first term is zero and occurs when

$$\underline{u}(0) = \underline{u}'(0) \qquad (3\text{-}31)$$

The matrix will be positive definite if R is positive definite, or if Q is positive definite and the columns of D are linearly independent. It will not be positive definite if R = 0, and the columns of D are linearly dependent. In other cases the matrix might be singular.

Equations (3-31), (3-28) and (3-29) define the optimal value of $\underline{u}(0)$.

The recurrence relation for P_{N+1} is determined by equating (3-30) and (3-23) when $\underline{u}(0) = \underline{u}'(0)$,

$$J^{o}_{N+1}\left[\underline{z}(0)\right] = \underline{z}^{T}(0)\phi^{T}(Q + P_{N})\phi\underline{z}(0)$$

$$- \underline{z}^{T}(0)A^{T}_{N+1}\left[D^{T}(Q + P_{N})D + R\right]A_{N+1}\underline{z}(0)$$

$$= \underline{z}^{T}(0)P_{N+1}\underline{z}(0) \qquad (3\text{-}32)$$

Since equation (3-32) must hold for all $\underline{z}(0)$, the recurrence relation becomes

$$P_{N+1} = \phi^T (Q + P_N) \, (\phi + DA_{N+1}) \qquad (3\text{-}33)$$

where

$$- A^T_{N+1} \left[D^T (Q + P_N) D + R \right] = \phi^T (Q + P_N) D \qquad (3\text{-}34)$$

or

$$A_{N+1} = - \left[D^T (Q + P_N) D + R \right]^{-1} D^T (Q + P_N) \phi \qquad (3\text{-}34a)$$

has been used to simplify (3-33).

Equation (3-32) shows that if the quadratic form for J^o_N is correct, then J^o_{N+1} has the same form. The quadratic form is trivially correct for J^o_0 since

$$J^o_0 \left[\underline{z}(0) \right] = 0 \qquad (3\text{-}35)$$

for all $\underline{z}(0)$. To complete the mathematical induction, the form for J^o_1 must be shown to be correct. J^o_1 is determined from equation (3-32) and (3-29) noting that $P_0 = 0$. Equation (3-34) is again used to simplify the result.

$$\begin{aligned} J^o_1 \left[\underline{z}(0) \right] &= \underline{z}^T(0) \phi^T Q \left[\phi - D(D^T Q D + R)^{-1} D Q \phi \right] \underline{z}(0) \\ &= \underline{z}^T(0) P_1 \underline{z}(0) \qquad (3\text{-}36) \end{aligned}$$

$J^o_1 \left[\underline{z}(0) \right]$ has the desired quadratic form.

3.4 Solution of the Problem

The solution proceeds using the recurrence relations, equations (3-33) and (3-34a). Since $J_0 \left[\underline{z}(0) \right] = 0$, $P_0 = 0$; calculate A_1. From A_1 and P_0 calculate P_1. Continue this process until all A_N of interest are calculated. If the plant is controllable, A_N will tend to a limit as N increases. Therefore, for the infinite stage regulator problem the optimal control takes the form of a stationary linear function of the states, which is independent of the sampling instants kT:

$$\underline{u}(k) = A_N \underline{z}(k) \tag{3-37}$$

However, if it is desired to optimize J_N over a finite number of samples, then the optimal control takes the form of a different linear function of the states at each sampling instant.

$$\underline{u}(k) = A(k)\underline{z}(k) \tag{3-38}$$

An advantage of this design procedure is that it always results in a stable control system. This is due to the state method of describing the system's dynamics, and to the use of dynamic programming to obtain the solution [3].

The resulting system is shown in Fig. 3-4. Note that it is a multiloop feedback system, requiring measurement of the plant's states as well as the input states.

If T becomes small, relative to the plant's time constants, the sampled signal $\underline{u}^*(t)$ is nearly the same as the continuous signal $\underline{u}(t)$. Then the sample and hold element can be removed from the system, resulting in a continuous system. If R = 0, and if T approaches zero, then the control input becomes unbounded. For this reason R must be greater than zero if T is going to be very small, or if a continuous system is desired.

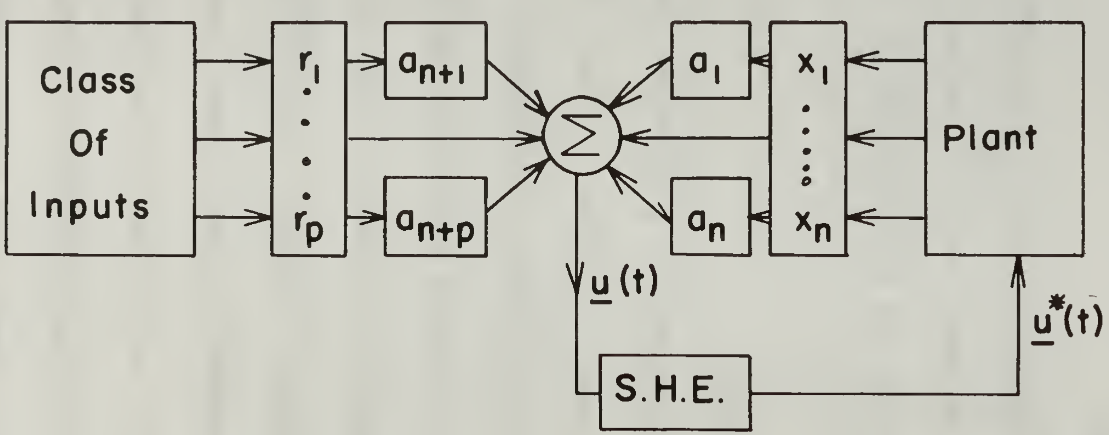

Fig. 3-4
SYSTEM DIAGRAM

A comment on notation is made here. When the control input $\underline{u}$ is a vector quantity the feedback control matrix will be denoted as A_N, whereas, if the control input is a scalar quantity, the feedback control matrix will be denoted as $\underline{a}_N^T$.

4. NUMERICAL EXAMPLES

To illustrate the control laws derived in Chapters 2 and 3, the following examples are presented. All simulations were done with the CDC 1604 digital computer.

4.1 Deadbeat Response

The third order system shown in Fig. 4-1 was simulated. The descriptive equations are

$$\underline{x}(k+1) = \phi \underline{x}(k) + \underline{d}u(k)$$

$$u(k) = \underline{a}_N^T \underline{x}(k)$$

The state transition matrix ϕ, and the $\underline{d}$ matrix for a sampling period $T = \pi/4$ are

$$\phi = \begin{bmatrix} 1 & .7091 & .2509 \\ 0 & .7265 & .5585 \\ 0 & -.6088 & .3914 \end{bmatrix}$$

$$\underline{d} = \begin{bmatrix} .06965 \\ .2509 \\ .5585 \end{bmatrix}$$

Using the deadbeat performance index

$$J_N = \underline{x}^T(N)\ I\ \underline{x}(N)$$

and the recurrence relations derived in Chapter 3, the following control law was computed, where, due to the use of a scalar control, N is equal to the order of the plant.

$$u(k) = \underline{a}_N^T \underline{x}(k) = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \underline{x}(k)$$

$$= \begin{bmatrix} -2.741672 & -3.127373 & -2.045130 \end{bmatrix} \underline{x}(k)$$

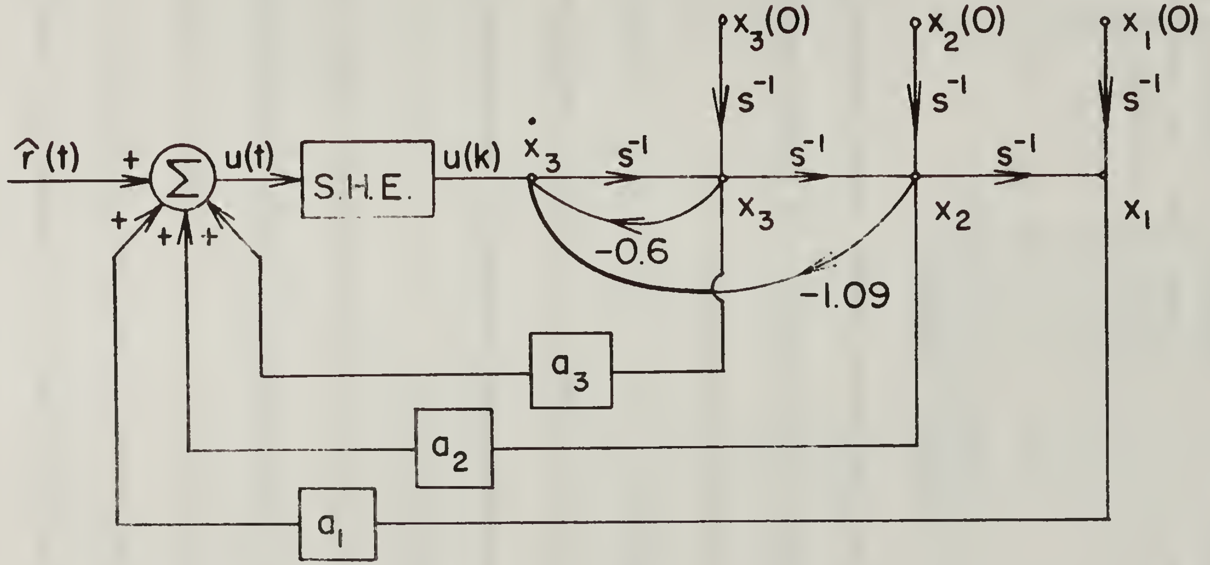

Fig. 4-1 THIRD ORDER SYSTEM

The same gain matrix was obtained using the methods of Chapter 2. The system's trajectory is presented in Table 4-1.

Stage(k)	Control u(k)	$x_1(k)$	$x_2(k)$	$x_3(k)$
Initial (0)	-2.7416	1.000000	0	0
(1)	3.064975	.808983	-.687893	-1.531268
(2)	-1.711838	.150569	-.585992	1.531268
Final (3)	0	0	0	0

TABLE 4-1 SYSTEM TRAJECTORY
FOR DEADBEAT EXAMPLE

The recurrence relations also yield a series of gain matrices forming a control law which is a different function of the states at each sampling instant,

$$u(k) = \underline{a}^T(k)\underline{x}(k)$$

This control law gives the identical control sequence u(0), u(1),...,u(N-1) as the stationary control law above.

4.2 Infinite Stage Regulator

The optimum control law for the infinite stage regulator is also in the form of a stationary linear function of the state variables. The control gains and the cost function can be computed by the recurrence relations of Chapter 3. After the optimum gains are established, the system's trajectory is computed from the difference equation of the system.

This example utilizes the same third order plant as the deadbeat example, Fig. 4-1. The problem was formulated as in Chapter 3 and the resulting system is as shown in Fig. 3-4. The performance index is

$$J_N = \sum_{k=1}^{N} \left[\underline{z}^T(k) Q \underline{z}(k) + u^2(k-1) R \right]$$

where

$$Q = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{array}{l} R = 1.0 \\ T = \pi/4 \end{array}$$

For a polynomial input such as described in equations (3-4) through (3-6), the gain matrix was computed to be

$$\underline{a}_N^T = \left[-.581355 \ -.626587 \ -.922714 \ .581355 \ 1.716587 \right]$$

The command inputs used were a unit step and a unit ramp, and the results are shown in Fig. 4-2 and 4-3 respectively. Both step and ramp responses were within 1% of the desired values at the end of eleven samples. Note that the gain matrix has the pattern expected for a system with one free integrator as discussed in Chapter 2; i.e., that $a_{n+1} = -a_1$ and $a_{n+2} = -(a_2 + d) = -(a_2 - 1.09)$. In the development of Chapter 2, it was shown that such a system would reach equilibrium with step and ramp command inputs only if in addition to the optimum plant state feedback gains, optimum gains on the command input states were incorporated; i.e., if the optimum gains are not applied to the input states, the system's ramp

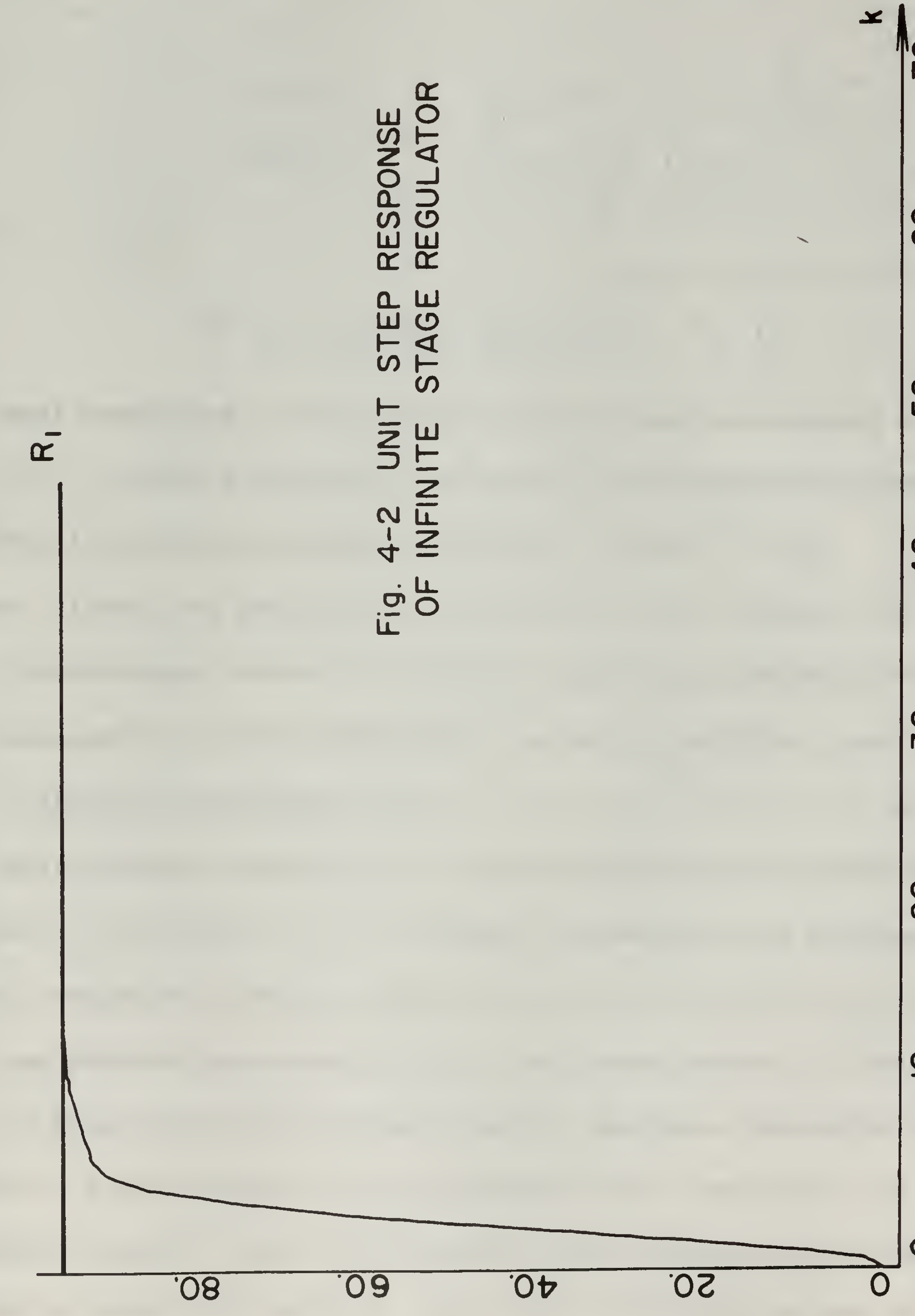

Fig. 4-2 UNIT STEP RESPONSE OF INFINITE STAGE REGULATOR

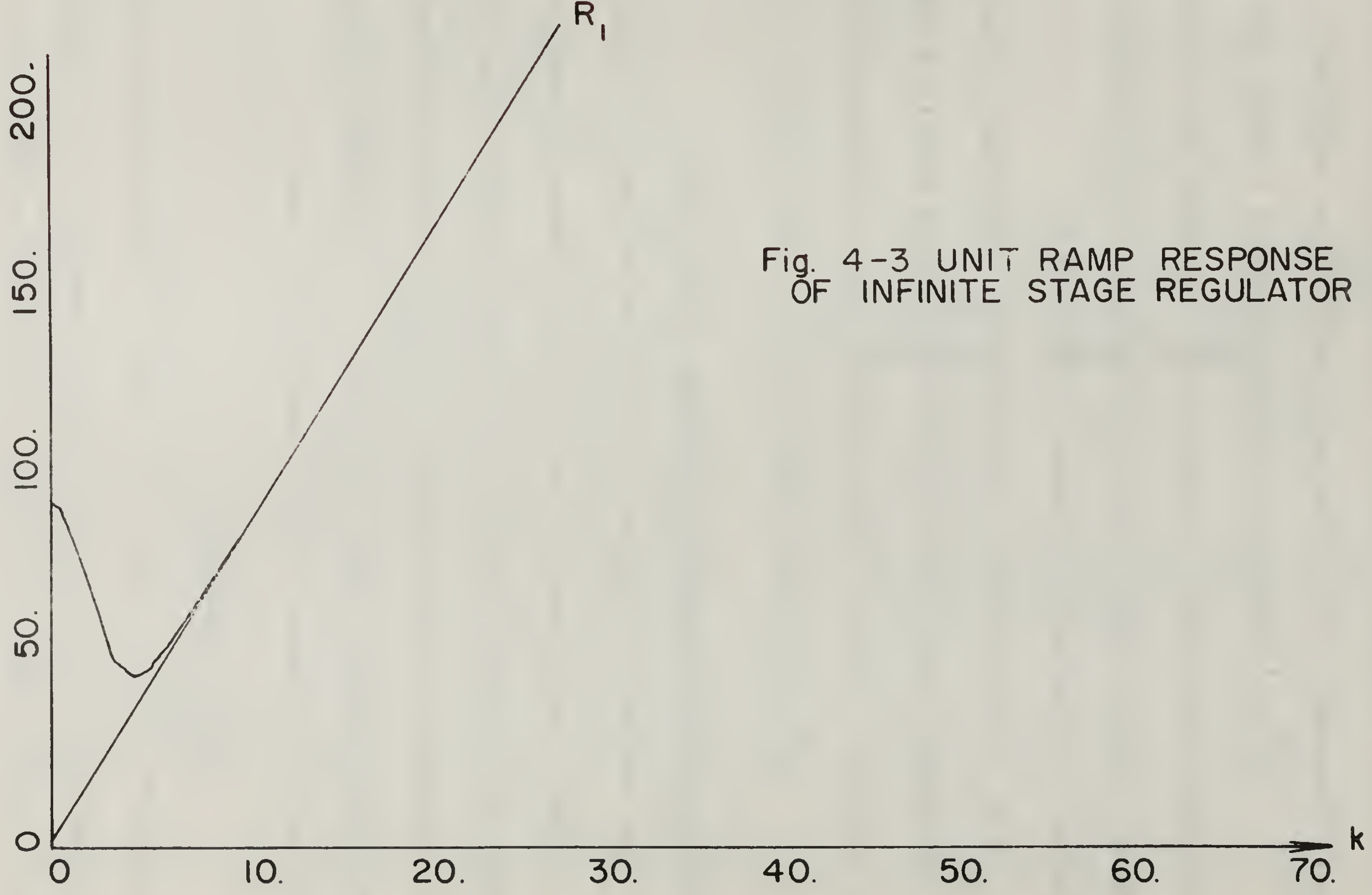

Fig. 4-3 UNIT RAMP RESPONSE OF INFINITE STAGE REGULATOR

response will have a steady state error. A ramp command input was applied to the example's system, without the gains on the input states, and the response is shown in Fig. 4-4.

For a sinusoidal command input as developed in equations (3-7) through (3-10), the gain matrix was computed to be

$$\underline{a}_N^T = \begin{bmatrix} -.581355 & -.626587 & -.922714 & .37783 & .581434 \end{bmatrix}$$

Note that the first n elements are the same as for the polynomial command input, but a_{n+1} and a_{n+2} are different. The system's trajectory is shown in Fig. 4-5.

Fig. 4-6 shows the trajectory for the same system with a sinusoidal input but with the weighting matrix Q as given by equation (3-15). The gain matrix was computed to be

$$\underline{a}_N^T = \begin{bmatrix} -4.484 & -4.275 & -2.313 & 4.080 & 1.815 \end{bmatrix}$$

4.3 The Finite Regulator Problem (CASE STUDY)

As stated in Chapter 3, if it is desired to optimize a system's performance over a finite number of samples M, where M is either smaller or not much greater than the number of recurrence steps required for A_N to tend to a limit, the resulting control takes the form of a different linear function of the states at each sampling instant,

$$\underline{u}(k) = A(k)\underline{x}(k)$$

A problem of this type would occur if the command input to the system were reinitialized every M samples. An example might be a missile that is using a search-while-track radar and is

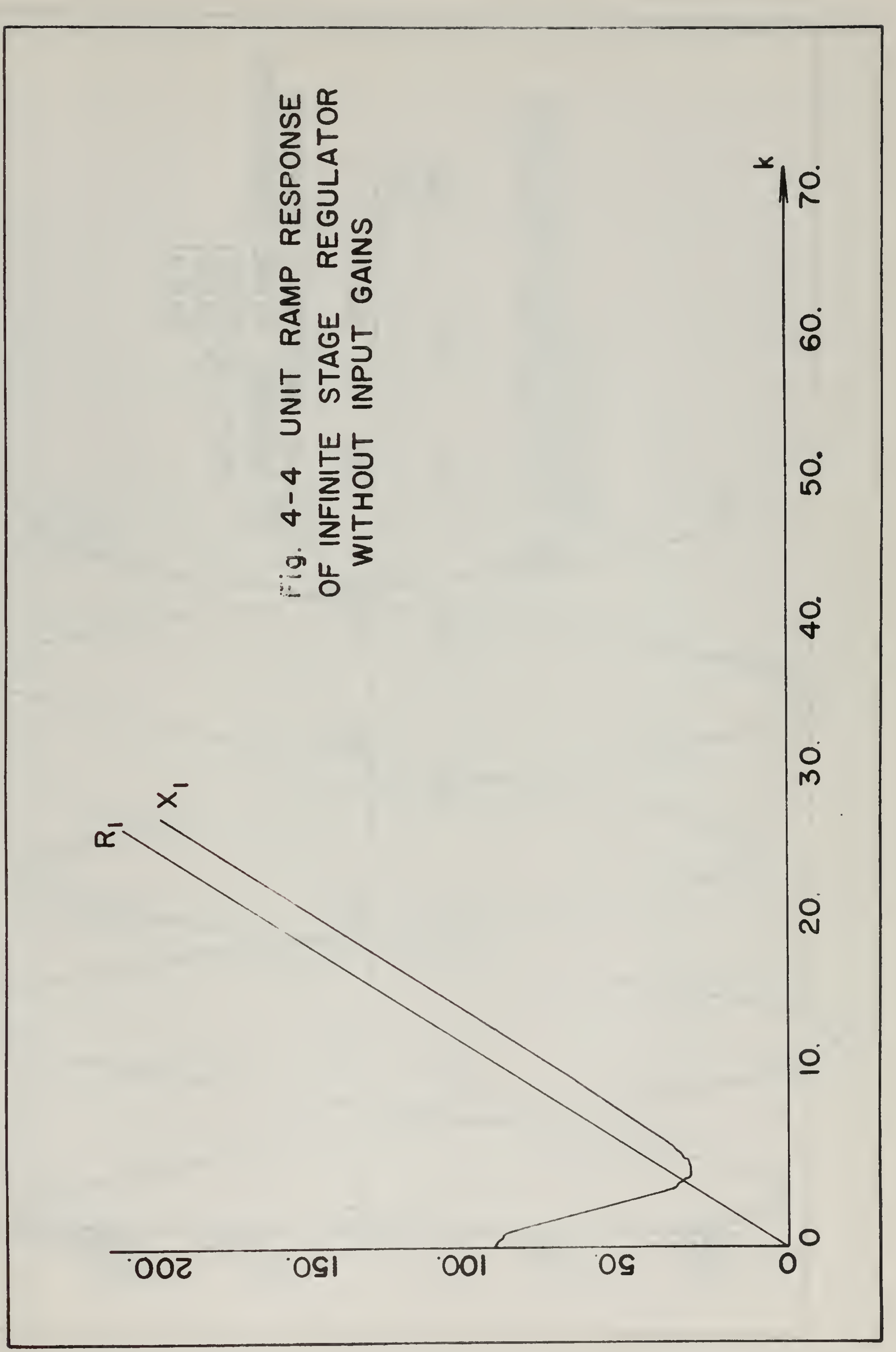

Fig. 4-4 UNIT RAMP RESPONSE OF INFINITE STAGE REGULATOR WITHOUT INPUT GAINS

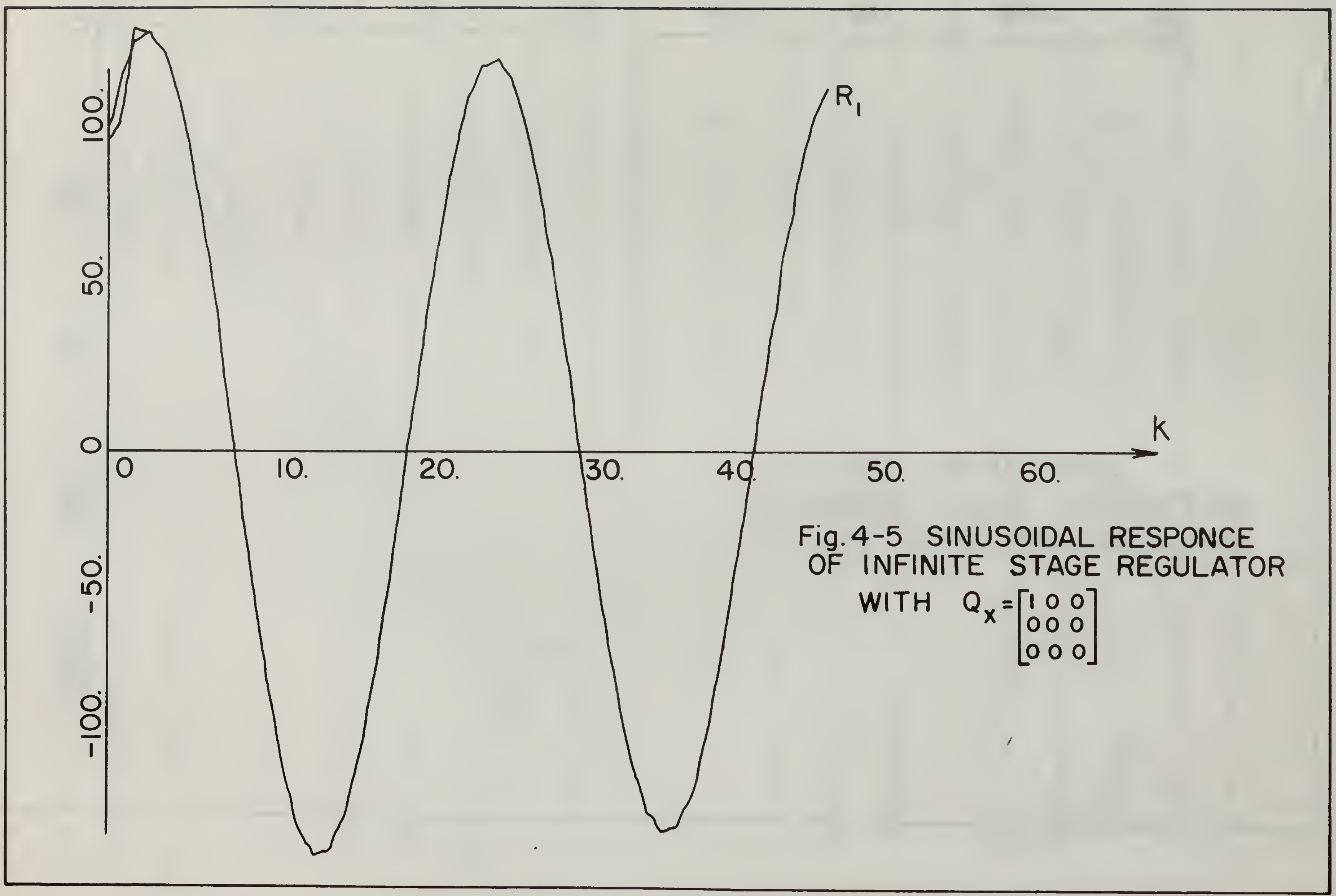

Fig. 4-5 SINUSOIDAL RESPONCE OF INFINITE STAGE REGULATOR WITH $Q_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

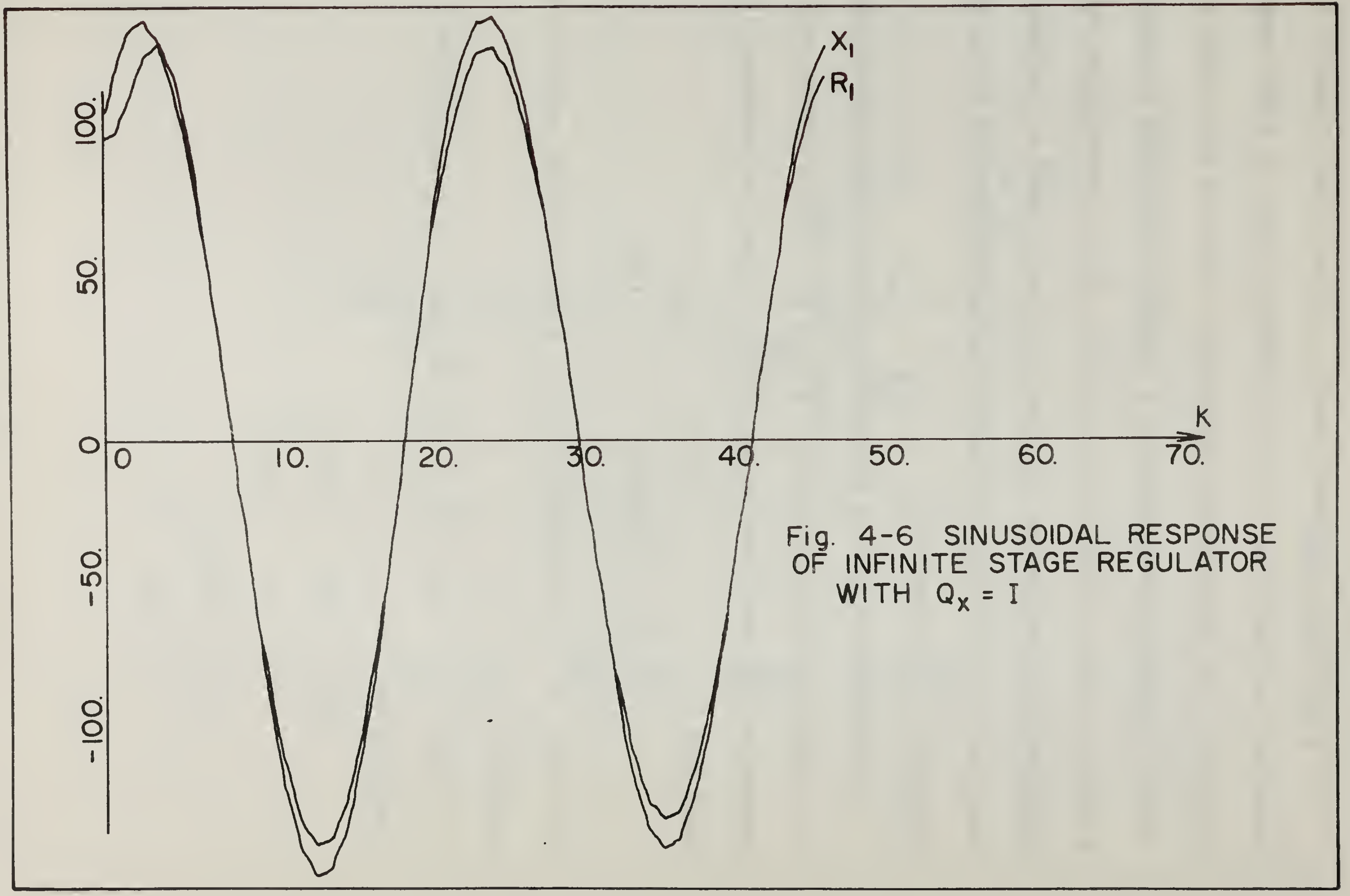

Fig. 4-6 SINUSOIDAL RESPONSE OF INFINITE STAGE REGULATOR WITH $Q_x = I$

receiving new information at the rate MT, where T is the system's sampling period.

To physically implement such a system would require a computer to reset the control feedback gains at each sampling instant. The engineering complexity involved with such a system warrants investigating a system with fixed control feedback gains. Though not optimal, this system might be considered superior due to other criteria such as weight, cost, etc. This form of control will be referred to as suboptimal in this section.

An investigation of suboptimal control was accomplished by simulating second, third and fourth order systems and comparing the performance to optimal. The performance index used was

$$J_N = \sum_{k=1}^{N} \left[\underline{x}^T(k) Q \underline{x}(k) + u^2(k-1) R \right]$$

and most of the investigation was made with the second order system of Fig. 4-7.

The state transition matrix ϕ, and the $\underline{d}$ matrix for a sampling period of 0.1 are

$$\phi = \begin{bmatrix} .9952 & .0931 \\ -.0931 & .8636 \end{bmatrix}$$

$$\underline{d} = \begin{bmatrix} .004768 \\ .0931 \end{bmatrix}$$

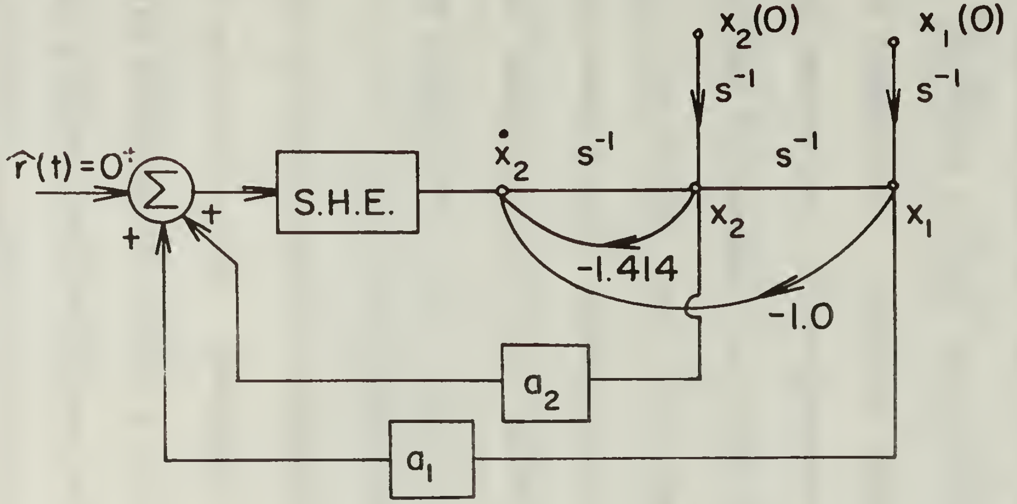

Fig. 4-7 SECOND ORDER SYSTEM

The weighting matrices are

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 1.0 \end{bmatrix}$$

The system was run for five samples.

It was observed that if the system was run for five samples using A_5 as the fixed control feedback matrix, the performance was not greatly degraded from that of optimal. This was also true of the higher order systems. The amount of performance degradation that is acceptable is, of course, an engineering decision based on the physical aspects of the problem.

Runs were also made using the values that A_N tended to in the limit. Even though these values gave optimal performance in the infinite stage regulator problem, the total cost was higher in the five-stage finite regulator problem than that obtained using the gain matrix A_5.

The elements of A_5 were then perturbed to see if the performance could be improved. This led to an exhaustive search which resulted in a gain matrix which differed significantly from A_5, and the resulting performance was near optimal. During the search the system's parameters and the initial conditions were held constant.

Using the gains obtained in the search, various initial conditions were tried, generally resulting in performance close to that obtained with A_5. The results of this case study are presented in Table 4-2. The column A_p contains the costs using gains obtained by perturbation.

Initial Conditions	Total Cost $J_5 = \sum_{1}^{5} \left[\underline{x}^T(k) I \underline{x}(k) + u^2(k-1) \right]$			
$\underline{x}(0)$	Optimal	Using A_5	Using A_∞	Using A_p
$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$	48.658	48.712	55.159	48.7714
$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *	8.0210	8.0500	9.5956	8.0214
$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$	28.0741	28.2912	34.2085	28.1898
$\begin{bmatrix} 10 \\ .2 \end{bmatrix}$	488.0655	488.2655	531.8718	490.7636
$\begin{bmatrix} .2 \\ 10 \end{bmatrix}$	230.8611	233.1031	270.0404	233.5962

TABLE 4-2 RESULTS OF CASE STUDY OF SUBOPTIMAL CONTROL FOR THE FINITE STAGE REGULATOR PROBLEM

* Initial conditions used during exhaustive search

Gain Matrices:

$$A_5 = \begin{bmatrix} -.038279 & -.264931 \end{bmatrix}$$

$$A = \begin{bmatrix} -.376449 & -.509776 \end{bmatrix}$$

$$A_p = \begin{bmatrix} .045198 & -.302418 \end{bmatrix}$$

Optimal (in order of application):

$$A_5 = \begin{bmatrix} -.038279 & -.264931 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} -.021133 & -.243223 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -.007999 & -.216953 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} .000587 & -.183544 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} .004439 & -.139468 \end{bmatrix}$$

5. CONCLUSIONS

5.1 Summary

5.1.1 Deadbeat Response

The optimum control law derived for deadbeat response takes the form of a stationary linear function of the state variables, independent of the sampling instant kT:

$$\underline{u}(k) = A_N \underline{x}(k)$$

where N is the number of stages required to bring the system to the origin, $N \leq n$.

It has also been shown that by using dynamic programming to minimize the performance index

$$J = \underline{x}^T(N) Q \underline{x}(N)$$

where $\underline{x}(N)$ is the final stage vector and Q is the identity matrix, the resulting control sequence will give deadbeat performance. The control law may be either stationary or a different linear function of the states at each sampling instant kT. The control sequence is the same in both cases.

$$\underline{u}(k) = A_N \underline{x}(k) = A(k) \underline{x}(k)$$

5.1.2 The Infinite Stage Regulator

The optimal control law derived for the infinite stage regulator is in the form of a stationary linear function of the state variables. The procedure for obtaining the constant control gains is well established [3, 7], and is given in Chapter 3.

5.1.3 The Finite Stage Regulator

The optimum control law derived for the finite stage regulator is a different linear function of the state variables at each sampling instant,

$$\underline{u}(k) = A(k)\underline{x}(k)$$

If it is desired to achieve the best performance possible with a stationary control law, the results of a case study tend to indicate that using A_M as the fixed gain matrix, where M is the number of sample periods, results in a system's performance not greatly degraded from optimum.

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APPENDIX I

Derivation of Difference Equations Representing a Sampled-Data Dynamic System

A block diagram of the system discussed in this Appendix is shown in Fig. 1-1.

Plant: A linear, time invariant dynamic system with state equations, which in vector notation may be written

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t) \qquad \text{(I-1)}$$

$$\underline{y}(t) = C\underline{x}(t) + D\underline{u}(t) \qquad \text{(I-2)}$$

The Laplace transformation of equation (I-1) is

$$s\underline{X}(s) - \underline{x}(0-) = A\underline{X}(s) + B\underline{U}(s) \qquad \text{(I-3)}$$

Rearranging,

$$s\underline{X}(s) - A\underline{X}(s) = \underline{x}(0-) + B\underline{U}(s)$$

$$(sI - A)\underline{X}(s) = \underline{x}(0-) + B\underline{U}(s)$$

$$\underline{X}(s) = (sI - A)\underline{x}(0-) + (sI - A)^{-1}B\underline{U}(s) \qquad \text{(I-4)}$$

Let

$$\phi(s) \triangleq (sI - A)^{-1}$$

$$\phi(t) \triangleq \mathcal{L}^{-1}[\phi(s)]$$

The solution of (I-1) becomes

$$\underline{x}(t) = \phi(t)\underline{x}(0-) + \int_0^t \phi(t-\tau)B\underline{u}(\tau)\,d\tau \qquad \text{(I-5)}$$

where it is assumed that $\underline{u}(t) = 0$; $t < 0$, and $\phi(t)$ is causal.

More generally: (initially $t = t_0$ vice zero)

$$\underline{x}(t) = \underbrace{\phi(t-t_0)\underline{x}(t_0)}_{\text{natural response}} + \underbrace{\int_{t_0}^{t} \phi(t-\tau)B\underline{u}(\tau)d\tau}_{\text{forced response}} \qquad \text{(I-6)}$$

For an interval of time T set:

$$t_0 = kT$$

$$t = (k+1)T$$

Then, (I-6) becomes:

$$\underline{x}((k+1)T) = \phi(T)\underline{x}(kT) + \int_{kT}^{(k+1)T} \phi((k+1)T-\tau)B\underline{u}(\tau)d\tau \qquad \text{(I-7)}$$

where $\underline{u}(t)$ is piecewise constant, i.e.,

$$\left.\begin{aligned} u_1(t) &= u_1(kT) \\ u_2(t) &= u_2(kT) \\ &\vdots \\ u_m(t) &= u_m(kT) \end{aligned}\right\} \text{for } kT \leq t < (k+1)T$$

Then,

$$\int_{kT}^{(k+1)T} \phi((k+1)T-\tau)B\underline{u}(\tau)d\tau = \left\{\int_0^T \phi(T-\tau)Bd\tau\right\}\underline{u}(kT)$$

$$= D(T)\underline{u}(kT)$$

Then, (I-7) becomes

$$\underline{x}(k+1)T = \phi(T)\underline{x}(kT) + D(T)\underline{u}(kT)$$

Some Properties of the State Transition Matrix

1) $\phi(0)$ is an identity matrix

2) $\phi(kT) = (\phi(T))^k$

Since by definition, $\phi(kT) = e^{ATk} = (e^{AT})^k$

3) $\phi(-kT) = (\phi(kT))^{-1}$

Since $\phi(-kT) = e^{-AkT} = (e^{AkT})^{-1}$

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DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)*
Naval Postgraduate School
Monterey, California 93940

2a. REPORT SECURITY CLASSIFICATION
UNCLASSIFIED

2b. GROUP

3. REPORT TITLE
OPTIMUM AND SUBOPTIMUM CONSTANT FEEDBACK CONTROL OF LINEAR SAMPLED-DATA SYSTEMS

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*
Master's Thesis in Aeroelectronics

5. AUTHOR(S) *(Last name, first name, initial)*
Howard, Phillip S., Lieutenant, U.S. Navy
Wasmund, Thomas Lee, Lieutenant, U.S. Navy

6. REPORT DATE June 1967	**7a. TOTAL NO. OF PAGES** 80	**7b. NO. OF REFS** 7

8a. CONTRACT OR GRANT NO.

b. PROJECT NO.

c.

d.

9a. ORIGINATOR'S REPORT NUMBER(S)

9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*

10. AVAILABILITY/LIMITATION NOTICES
[redacted]
This document has been approved for public release and sale; its distribution is unlimited.

11/25/69

11. SUPPLEMENTARY NOTES

12. SPONSORING MILITARY ACTIVITY
Naval Air Systems Command

13. ABSTRACT

In this paper constant feedback control laws are investigated for linear sampled-data systems. Derivations for the minimum-time problem and the infinite stage regular problem which yield constant feedback optimal control laws are presented. A case study for the finite stage regulator problem indicates that constant feedback control, although suboptimal, may be feasible from a design viewpoint. Several numerical examples illustrate the results. (U)

DD FORM 1 JAN 64 1473

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
1. Optimum Control						
2. Sampled-data Control						
3. Discrete Systems						
4. Feedback Control						
5. Constant Feedback Control						

[redacted]

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